Applied Newton-Cartan Geometry

Eric Bergshoeff

Groningen University

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rijksuniversiteit groningen

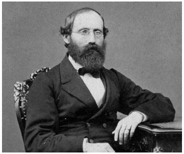
Newtonian Gravity

- Free-falling frames: Galilean symmetries

- Earth-based frame: Newton potential

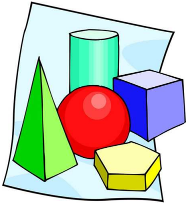
- <u>no</u> frame-independent formulation

Geometry



Riemann (1867)

for a frame-independent formulation you need geometry



Einstein 1915

Einstein achieved two things in 1915:

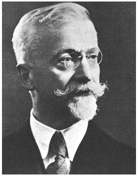
- He made gravity consistent with special relativity
- He used a frame-independent formulation

General Relativity

- Free-falling frames: Poincare symmetries

- arbitrary frames: metric tensor field

Newton-Cartan Gravity



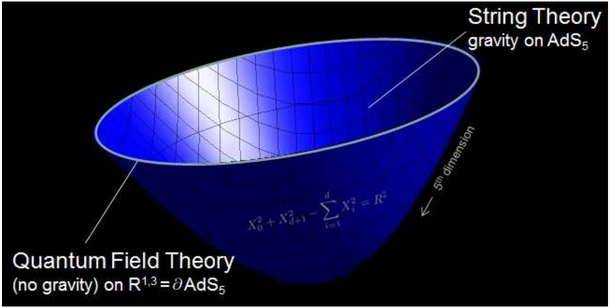
Elie Cartan 1923

- Newton-Cartan (NC) gravity is Newtonian gravity in arbitrary frame
- NC gravity contains more gravitational fields than Newton potential
- Newtonian gravity is re-obtained by gauge-fixing

Newton-Cartan Geometry

NC geometry is foliated with an absolute time why non-relativistic gravity ?

The Holographic Principle



In special cases one finds NC gravity with twistless torsion

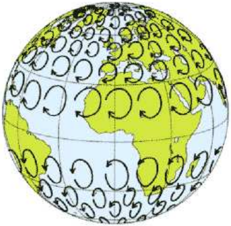
Christensen, Hartong, Kiritsis, Obers and Rollier (2013-2015)

Condensed Matter Physics

Effective Field Theory (EFT) coupled to NC gravity ⇒ universal features

Examples: fractional quantum Hall effect, chiral superfluids, simple fluids

compare to



Coriolis force

Son (2005, 2012), Can, Laskin, Wiegmann (2014), Jensen (2014), Gromov, Abanov (2015)

Outline

Newton-Cartan from gauging Bargmann

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Newton-Cartan from gauging Bargmann

Adding Matter

Outline

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Newton-Cartan from gauging Bargmann

Adding Matter

Special Case: 3 Space-time Dimensions

Future Directions

Einstein Gravity

In the relativistic case free-falling frames are connected by the Poincare symmetries:

- space-time translations: $\delta x^\mu = \xi^\mu$
- Lorentz transformations: $\delta x^\mu = \lambda^\mu{}_\nu \, x^\nu$

in arbitrary frames the gravitational force is described by an invertable Vierbein field $e_\mu{}^A(x)$

$\mu = 0, 1, 2, 3$; A=0,1,2,3

Non-relativistic Gravity

In the non-relativistic case free-falling frames are connected by the

Galilean symmetries:

- time translations: $\delta t = \xi^0$
- space translations: $\delta x^i = \xi^i \qquad i = 1, 2, 3$
- spatial rotations: $\delta x^i = \lambda^i{}_j\, x^j$
- Galilean boosts: $\delta x^i = \lambda^i t$

Newtonian gravity versus Newton-Cartan gravity

- in frames with constant acceleration $(\delta x^i = \frac{1}{2}a^i t^2)$ the gravitational force is described by the Newton potential $\Phi(\vec{x}) \rightarrow$

Newtonian gravity

- in arbitrary frames the gravitational force is described by a temporal Vierbein $\tau_\mu(x)$, spatial Vierbein $e_\mu{}^a(x)$ <u>plus</u> a vector $m_\mu(x) \rightarrow$

$\mu = 0, 1, 2, 3$; a=1,2,3

Newton-Cartan (NC) gravity

Relativistic versus Non-relativistic Particle

$$S_{\rm relativistic} = -m \int d\tau \, \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \qquad \mu = 0, 1, 2, 3$$

Lagrangian is invariant under Poincare symmetries

$$S_{\rm non\text{-}relativistic} = \frac{m}{2} \int \frac{\dot{x}^i \dot{x}^j \delta_{ij}}{\dot{t}} d\tau \qquad i = 1, 2, 3$$

Lagrangian is <u>not</u> invariant under Galilean boosts:

$$\delta L_{\rm \ non\text{-}relativistic} = \frac{d}{d\tau}(m x^i \lambda^j \, \delta_{ij}) \qquad \Rightarrow$$

modified Noether charge gives rise to central extension!

'Gaugings', Contractions and Non-relativistic Limits

$$\text{Poincare} \otimes \text{U(1)} \quad \overset{\text{'gauging'}}{\Longrightarrow} \quad \text{General relativity} \otimes \text{U(1)}$$

contraction $\Downarrow$ $\qquad\qquad$ $\Downarrow$ non-relativistic limit

$$\text{Bargmann} \quad \overset{\text{'gauging'}}{\Longrightarrow} \quad \text{Newton-Cartan gravity}$$

'Gauging' the Bargmann algebra

Andringa, Panda, de Roo + E.B. (2011); cp. to Chamseddine and West (1977)

$$[J_{ab}, P_c] = -2\delta_{c[a}P_{b]}\,, \qquad [J_{ab}, G_c] = -2\delta_{c[a}G_{b]}\,,$$

$$[G_a, H] = -P_a\,, \qquad [G_a, P_b] = -\delta_{ab}Z\,, \qquad a = 1, 2, \ldots, d$$

symmetry	generators	gauge field	parameters	curvatures
time translations	H	τ_μ	$\zeta(x^\nu)$	$\mathcal{R}_{\mu\nu}(H)$
space translations	P^a	${e_\mu}^a$	$\zeta^a(x^\nu)$	${\mathcal{R}_{\mu\nu}}^a(P)$
Galilean boosts	G^a	${\omega_\mu}^a$	$\lambda^a(x^\nu)$	${\mathcal{R}_{\mu\nu}}^a(G)$
spatial rotations	J^{ab}	${\omega_\mu}^{ab}$	$\lambda^{ab}(x^\nu)$	${\mathcal{R}_{\mu\nu}}^{ab}(J)$
central charge transf.	Z	m_μ	$\sigma(x^\nu)$	$\mathcal{R}_{\mu\nu}(Z)$

Imposing Constraints

$\mathcal{R}_{\mu\nu}{}^{a}(P) = 0\,, \qquad \mathcal{R}_{\mu\nu}(Z) = 0\ :$ solve for spin-connection fields

$\mathcal{R}_{\mu\nu}(H) = \partial_{[\mu}\tau_{\nu]} = 0\ \to\ \tau_\mu = \partial_\mu\,\tau\ :$ absolute time ('zero torsion')

$\mathcal{R}_{\mu\nu}{}^{ab}(J) \neq 0\ :$ un-constrained off-shell

$\mathcal{R}_{0(a,b)}(G) \neq 0\ :$ un-constrained off-shell

The Transformation Rules

The independent NC fields $\{\tau_\mu, e_\mu{}^a, m_\mu\}$ transform as follows:

$$\begin{aligned}
\delta\tau_\mu &= \xi^\lambda\partial_\lambda\tau_\mu + \partial_\mu\xi^\lambda\tau_\lambda\,,\\
\delta e_\mu{}^a &= \xi^\lambda\partial_\lambda e_\mu{}^a + \partial_\mu\xi^\lambda e_\lambda{}^a + \lambda^a{}_b\, e_\mu{}^b + \lambda^a\tau_\mu\,,\\
\delta m_\mu &= \xi^\lambda\partial_\lambda m_\mu + \partial_\mu\xi^\lambda m_\lambda + \partial_\mu\sigma + \lambda_a\, e_\mu{}^a
\end{aligned}$$

The spin-connection fields $\omega_\mu{}^{ab}$ and $\omega_\mu{}^a$ are functions of e, τ and m

There are two Galilean-invariant metrics:

$$\tau_{\mu\nu} = \tau_\mu\tau_\nu\,, \qquad h^{\mu\nu} = e^\mu{}_a e^\nu{}_b\,\delta^{ab}$$

from general frames back to special frames

free-falling frames: $\tau_\mu = \delta_\mu^t\,, \qquad e_t{}^a = 0, \quad e_i{}^a = \delta_i^a\,, \qquad m_\mu = 0$

$$\Downarrow$$

Non-relativistic Killing equations

$$\partial_\mu \xi^t = 0\,, \qquad \partial_t \xi^i + \lambda^i = 0\,,$$
$$\partial_i \xi^j + \lambda^j{}_i = 0\,, \qquad \partial_t \sigma = 0\,, \qquad \partial_i \sigma + \lambda_i = 0$$

$$\Downarrow$$

Galilean Symmetries

$$\xi^t(x^\mu) = \zeta\,, \qquad \xi^i(x^\mu) = \xi^i - \lambda^i\, t - \lambda^i{}_j\, x^j\,, \qquad \sigma(x^\mu) = \sigma - \lambda^i\, x^i$$

The NC Equations of Motion

The NC equations of motion are given by

$$\begin{aligned} \tau^{\mu} e^{\nu}{}_{a} \mathcal{R}_{\mu\nu}{}^{a}(G) &= 0 \qquad \mathbf{1} \\ e^{\nu}{}_{a} \mathcal{R}_{\mu\nu}{}^{ab}(J) &= 0 \qquad \mathbf{a}, (\mathbf{ab}) \end{aligned}$$

- after gauge-fixing and assuming flat space the first NC e.o.m. becomes $\triangle\Phi = 0$
- note: there is no action that gives rise to these equations of motion

Outline

How do we add matter?

We obtain matter couplings from arbitrary contracting backgrounds

Rosseel, Zojer + E.B. (2015)

$$\text{Klein-Gordon} \quad \overset{?}{\Longrightarrow} \quad \text{Schrödinger}$$

$$\text{general frames} \Downarrow \qquad\qquad \Uparrow \text{free-falling frames}$$

$$\text{scalar} + \text{GR} \quad \overset{\text{`limit'}}{\Longrightarrow} \quad \text{Schrödinger} + \text{NC}$$

Inönü Wigner Contraction

$$\left[P_A, M_{BC}\right] = 2\,\eta_{A[B}\,P_{C]}\,, \qquad \left[M_{AB}, M_{CD}\right] = 4\,\eta_{[A[C}\,M_{D]B]} \qquad \text{plus} \qquad \mathcal{Z}$$

$$P_0 = \frac{1}{2\omega}\,H + \omega\,Z\,, \qquad \mathcal{Z} = \frac{1}{2\omega}\,H - \omega\,Z\,, \qquad A = (0, a)$$

$$P_a = P_a\,, \qquad M_{ab} = J_{ab}\,, \qquad M_{a0} = \omega\,G_a$$

Taking the limit $\omega \to \infty$ gives the Bargmann algebra including Z:

$$\left[P_a, G_b\right] = \delta_{ab}\,Z$$

The Non-relativistic Limit I

Dautcourt (1964), Jensen, Karch (2014), Rosseel, Zojer + E.B. (2015)

add a vector field M_μ to general relativity with $\partial_{[\mu} M_{\nu]} = 0$

STEP I: make field redefinitions $\{E_\mu{}^A, M_\mu\} \rightarrow \{\tau_\mu, e_\mu{}^a, m_\mu\}$

$$E_\mu{}^0 = \omega\,\tau_\mu + \frac{1}{2\omega}\,m_\mu\,, \quad M_\mu = \omega\,\tau_\mu - \frac{1}{2\omega}\,m_\mu\,, \quad E_\mu{}^a = e_\mu{}^a \quad \Rightarrow$$

$$E^\mu{}_a = e^\mu{}_a - \frac{1}{2\omega^2}\,\tau^\mu e^\rho{}_a m_\rho + \mathcal{O}\left(\omega^{-4}\right) \text{ and similar for } E^\mu{}_0$$

The Non-relativistic Limit II

STEP II: take the limit $\omega \to \infty$ $\Rightarrow$

- the correct transformation rules are obtained
- the correct equations of motion are obtained (but no action!)

Note: the standard textbook limit gives Newton gravity

From Klein-Gordon to Schrödinger I

we consider a complex scalar field with mass M

Lévy Leblond (1963,1967)

$$E^{-1}\,\mathcal{L}_{\rm rel} = -\frac{1}{2}\,g^{\mu\nu}\,D_\mu\Phi^*D_\nu\Phi - \frac{M^2}{2}\,\Phi^*\Phi \quad \text{with}$$

$$D_\mu\Phi = \partial_\mu\Phi - \mathrm{i}\,M\,M_\mu\,\Phi\,, \qquad \delta\Phi = \mathrm{i}\,M\,\Lambda\,\Phi$$

- M_μ is not an electromagnetic field ($M \neq q$)!
- M_μ couples to the current that expresses conservation of # particles – # antiparticles
- going to free-falling frames gives Klein-Gordon

From Klein-Gordon to Schrödinger II

make redefinitions plus $M = \omega m\,, \Phi \to \sqrt{\frac{\omega}{m}}\Phi$ and take limit $\omega \to \infty \quad \to$

$$e^{-1}\mathcal{L}_{\text{Schroedinger}} = \Big[\frac{\mathrm{i}}{2}\Big(\Phi^* \tilde{D}_0\Phi - \Phi\tilde{D}_0\Phi^*\Big) - \frac{1}{2m}\Big|\tilde{D}_a\Phi\Big|^2\Big] \quad \text{with}$$

$$\tilde{D}_\mu\Phi = \partial_\mu\Phi + \mathrm{i}\, m\, m_\mu\,\Phi\,, \qquad \delta\Phi = \xi^\mu\partial_\mu\Phi - \mathrm{i}\, m\,\sigma\,\Phi$$

- m_μ couples to the current that expresses conservation of # particles
- going to free-falling frames gives Schrödinger

Note: Bargmann is extended to Schrödinger

Outline

Newton-Cartan from gauging Bargmann

Adding Matter

Special Case: 3 Space-time Dimensions

Future Directions

Non-relativistic Limit of Einstein plus Scalar

$$S = \int d^D x \sqrt{-g} \left\{ \frac{1}{\kappa^2} R - \frac{1}{2} g^{\mu\nu} D_\mu \Phi^* D_\nu \Phi - \frac{M^2}{2} \Phi^* \Phi \right\} \quad \Rightarrow$$

$$R_{\mu\nu} = \kappa^2 \left(T_{\mu\nu} - \frac{1}{D-2} g_{\mu\nu} T \right)$$

Taking the non-relativistic limit plus $\kappa^2 \to \kappa^2/\omega^4$ gives

$$\begin{aligned} R_{0a}{}^a(G) &= \frac{D-3}{D-2} \kappa^2 \rho \quad \text{with} \quad \rho = m \Phi^* \Phi \\ R_{0b}{}^b{}_a(J) &= R_{ac}{}^c{}_b(J) = 0 \end{aligned} \quad \stackrel{D=4}{\Longrightarrow}$$

$$\triangle \Phi = 4\pi G_N \rho$$

Exotic Galilei Symmetries

Papageorgiou, Schroers (2009); Rosseel + E.B., work in progress

Galilei $\overset{\text{`Mass'}}{\Longrightarrow}$ Bargmann $\overset{\text{`Spin'}}{\Longrightarrow}$ Exotic Galilei

$$[J_A, P_B] = -\epsilon_{ABC} P^C\,, \qquad [J_A, J_B] = -\epsilon_{ABC} J^C \qquad \text{plus} \qquad \mathcal{Z}_1\,, \mathcal{Z}_2$$

$$P_0 = \frac{1}{2\omega} H + \omega M\,, \qquad J_0 = \frac{1}{2} J + \omega^2 S\,, \qquad J_a = \omega G_a\,,$$

$$\mathcal{Z}_1 = \frac{1}{2\omega} H - \omega M\,, \qquad \mathcal{Z}_2 = \frac{1}{2} J - \omega^2 S \qquad \omega \to \infty \quad \Rightarrow$$

$$[H, G_a] = -\epsilon_{ab} P_b\,, \qquad [J, G_a] = -\epsilon_{ab} G_b\,, \qquad [J, P_a] = -\epsilon_{ab} P_b\,,$$

$$[G_a, G_b] = \epsilon_{ab} S\,, \qquad [G_a, P_b] = \epsilon_{ab} M$$

3D Non-relativistic Chern-Simons Gravity

3D exotic Galilei has invariant, non-degenerate bilinear form:

$$(J_a, P_b) = \delta_{ab}\,, \qquad (M, J) = -1\,, \qquad (H, S) = -1 \qquad \Rightarrow$$

$$S = \frac{k}{4\pi}\int d^3x \left(\epsilon^{\mu\nu\rho} e_\mu{}^a R_{\nu\rho}{}^a(G) - \epsilon^{\mu\nu\rho} m_\mu R_{\nu\rho}(J) - \epsilon^{\mu\nu\rho}\tau_\mu R_{\nu\rho}(S)\right)$$

E.O.M.

$$\omega_\mu, \omega_\mu{}^a : \qquad R_{\mu\nu}{}^a(P) = 0\,, \qquad R_{\mu\nu}(Z) = 0\,,$$

$$e_\mu{}^a\, \tau_\mu\,, m_\mu : \qquad R_{\mu\nu}{}^a(G) = 0\,, \qquad R_{\mu\nu}(J) = 0\,, \qquad R_{\mu\nu}(S) = 0\,,$$

$$s_\mu : \qquad R_{\mu\nu}(H) = \partial_{[\mu}\tau_{\nu]} = 0$$

3D Non-Relativistic Limit

$$(J_A, P_B) = \eta_{AB}\,, \qquad (\mathcal{Z}_1, \mathcal{Z}_2) = 1 \qquad \Rightarrow$$

$$S = \frac{k}{4\pi}\int d^3x\Big(2\epsilon^{\mu\nu\rho}E_\mu{}^A\partial_\nu\Omega_{\rho A} - \epsilon^{\mu\nu\rho}\epsilon_{ABC}E_\mu{}^A\Omega_\nu{}^B\Omega_\rho{}^C + 2\epsilon^{\mu\nu\rho}M_\mu\partial_\nu S_\rho\Big)$$

$$E_\mu{}^0 = \omega\tau_\mu + \frac{1}{2\omega}m_\mu\,, \qquad M_\mu = \omega\tau_\mu - \frac{1}{2\omega}m_\mu$$

$$\Omega_\mu{}^0 = \omega_\mu + \frac{1}{2\omega^2}s_\mu\,, \qquad S_\mu = \omega_\mu - \frac{1}{2\omega^2}s_\mu$$

$$E_\mu{}^a = e_\mu{}^a\,, \qquad \Omega_\mu{}^a = \frac{1}{\omega}\omega_\mu{}^a$$

plus $k \to k\omega$

Adding Matter and Comparing to Newton-Cartan

couple both theories to Klein-Gordon $\Rightarrow$

Newton-Cartan

- $R_{00} \sim j^0$ and all other Ricci-tensor components are zero
- $R_{\mu\nu}(J) = 0$

3D non-relativistic CS

- $R_{00} \sim t_a{}^a$ plus more equations involving the other currents
- $R_{\mu\nu}(J) \neq 0$: more general backgrounds!

Outline

Applications and Extensions

- extend from ‘particle’ limits to ‘stringy’ limits

- relation to Hořava-Lifshitz gravity

  Hartong and Obers (2015)

- non-relativistic conformal method and Schrödinger symmetries

  Afshar, Mehra, Parekh, Rollier + E.B. (2015)

Supersymmetry

(I) the supersymmetric extension of General Relativity was discovered precisely 40 years ago Freedman, Ferrara and van Nieuwenhuizen but today there is no known supersymmetric extension of Newton's Poisson equation

Freedman, Ferrara, van Nieuwenhuizen (1976)

(II) (non-relativistic) supersymmetry allows to apply powerful localization techniques to extract exact results

Pestun (2007)

Applications to Condensed Matter Models!

March 6-10, 2017: Save the Date!

Simons Workshop on Applied Newton-Cartan Geometry

organized by Gary Gibbons, Rob Leigh, Djordje Minic, Dam Thanh Son + E.B.