The Hunting of the Star-product

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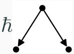
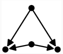
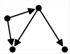
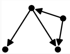
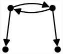
Supervisor: Arthemy Kiselev (September 2014 – now)

University of Groningen, The Netherlands

Symposium on advances in semi-classical methods in mathematics and physics

$\bullet \star \bullet = \bullet\, \bullet + \hbar$  $+ \hbar^2 \Big(\frac{1}{2}$  $+ \frac{1}{3}$  $+ \frac{1}{3}$  $+ \frac{1}{6}$  $\Big) + \mathcal{O}(\hbar^3).$

Deformation quantization

Groenewold's star-product

See *On the principles of elementary quantum mechanics*, p. 48.

- Let $\mathbb{R}^{2n}$ be phase space with coordinates (q, p).
- Observables live in $C^\infty(\mathbb{R}^{2n}) = \{f : \mathbb{R}^{2n} \to \mathbb{R} : f \text{ smooth}\}$.
- Classical Poisson bracket $\{\ ,\ \}$ on $C^\infty(\mathbb{R}^{2n})$:

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).$$

- Groenewold displays a "quantum product" $\star$ such that

$$f \star g = fg + O(\hbar) \quad \text{and} \quad f \star g - g \star f = i\hbar\{f, g\} + O(\hbar^2).$$

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For convenience, put $\Pi = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$ and $\partial_i = \partial/\partial x_i$ for $x = (q, p)$.

Formula: $f \star g = fg + \frac{i\hbar}{2}\Pi^{ij}\partial_i(f)\partial_j(g) - \frac{\hbar^2}{4}\Pi^{ij}\Pi^{kl}\partial_i\partial_k(f)\partial_j\partial_l(g) + \dots$

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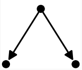
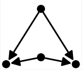
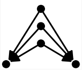
Preview: $\bullet \star \bullet = \bullet\,\bullet + \frac{i\hbar}{2}$  $- \frac{\hbar^2}{4}$  $- \frac{i\hbar^3}{8}$  $+ \mathcal{O}(\hbar^4)$

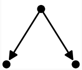
Review: Poisson brackets

Definition

A Poisson bracket $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ on a smooth manifold M is a skew-symmetric bi-derivation satisfying the Jacobi identity:

$$\begin{aligned} \{f,g\} &= -\{g,f\}, \\ \{f,g+h\} &= \{f,g\} + \{f,h\}, \\ \{f,gh\} &= \{f,g\}h + g\{f,h\}, \\ \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} &= 0 \\ \text{for all } f,g,h &\in C^\infty(M) \end{aligned}$$

In coordinates: $\{f,g\} = \Pi^{ij}\partial_i(f)\partial_j(g) = (f) \xleftarrow{i} \Pi^{ij} \xrightarrow{j} (g)$ for skew-symmetric matrix of functions Π^{ij}.



Deformation quantization of Poisson manifolds

Given

- $M \leftarrow$ smooth Poisson manifold with bracket $\{\ ,\ \}$.
- $C^\infty(M) \leftarrow$ smooth scalar functions on M, form a Poisson algebra under $\{\ ,\ \}$ and a commutative associative unital $\mathbb{R}$-algebra under $(fg)(x) = f(x)g(x)$.

Can we *deform* the product into $\star : A \times A \to A$ given by

$$f \star g = fg + \hbar B_1(f,g) + \hbar^2 B_2(f,g) + \ldots \qquad \text{for } f,g \in C^\infty(M),$$

while staying unital, **associative**, and such that

$$f \star g - g \star f = \hbar\{f,g\} + O(\hbar^2)?$$

Here the B_i are bi-{linear, differential} operators; $A = C^\infty(M)[\![\hbar]\!]$.

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More wishful version: can we have $f \star g = fg + \hbar\{f,g\} + \mathcal{O}(\hbar^2)$?

Deformation quantization of Poisson manifolds, I

M. Kontsevich (1997): Yes we can!

Explicitly, on an affine manifold, let $\{f, g\} = \sum_{i,j} \Pi^{ij}\partial_i f\,\partial_j g$,

Then a star product is given (up to order 2) by

$$f \star g = fg + \hbar\Pi^{ij}\partial_i f\,\partial_j g + \hbar^2\left(\tfrac{1}{2}\Pi^{ij}\Pi^{kl}\partial_i\partial_k f\,\partial_j\partial_l g + \tfrac{1}{3}\Pi^{ij}\partial_j\Pi^{kl}\partial_i\partial_k f\,\partial_l g + \tfrac{1}{3}\Pi^{ij}\partial_j\Pi^{kl}\partial_k f\,\partial_i\partial_l g\right) + \mathcal{O}(\hbar^3).$$

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Explicit *universal* formula in terms of (derivatives of) Poisson bi-vector components Π^{ij}, at all orders in $\hbar$.

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Where does it come from? Sum of weighted graphs:

$$f\star g = fg + \sum_{n=1}^{\infty}\frac{\hbar^n}{n!}\left(\sum_{\Gamma\in G_n} w(\Gamma)B_\Gamma(f,g)\right)$$

Graphs and weights

Graphs

Kontsevich graphs in G_n ($n \geq 1$):

- *n internal* vertices; each has two *ordered* outgoing edges,
- two ordered *ground* vertices without outgoing edges,
- no tadpoles.

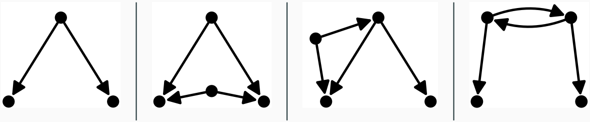

Table 1: Some Kontsevich graphs ($n \leq 2$).

NB: Ordering of edges in pictures is Left $\prec$ Right.

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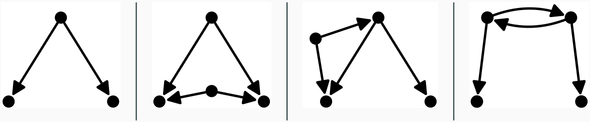

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Operators

Recipe for bi-differential operator B_Γ associated to graph Γ:

- label edges from internal vertex with indices, say i, j,
- substitute Poisson bi-vector component Π^{ij} into the vertex,
- substitute arguments f, g into ground vertices,
- incoming edge i acts as derivative ∂_i on target vertex,
- multiply (differentiated) contents of vertices,
- sum over all indices.

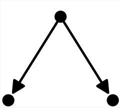
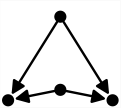
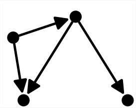
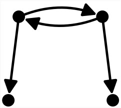
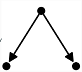
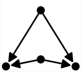
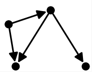
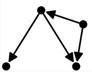
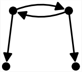
			
$\Pi^{ij}\ \partial_i \otimes \partial_j$	$\Pi^{ij}\Pi^{kl}\ \partial_i\partial_k \otimes \partial_j\partial_l$	$\Pi^{ij}\partial_j\Pi^{kl}\ \partial_i\partial_k \otimes \partial_l$	$\partial_k\Pi^{ij}\partial_j\Pi^{kl}\ \partial_i \otimes \partial_l$

Table 2: Some bi-differential operators associated to graphs.

Star product

Kontsevich's star product is given up to $\mathcal{O}(\hbar^2)$ by

$\bullet \star \bullet = \bullet\bullet + \hbar$  $+$

$\hbar^2 \Big(\frac{1}{2}$  $+ \frac{1}{3}$  $+ \frac{1}{3}$  $+ \frac{1}{6}$  $\Big)$

$+ O(\hbar^3).$

$$f \star g = fg + \hbar \Pi^{ij} \partial_i f \, \partial_j g + \hbar^2 \big(\tfrac{1}{2} \Pi^{ij} \Pi^{kl} \partial_i \partial_k f \, \partial_j \partial_l g + \tfrac{1}{3} \Pi^{ij} \partial_j \Pi^{kl} \partial_i \partial_k f \, \partial_l g + \tfrac{1}{3} \Pi^{ij} \partial_j \Pi^{kl} \partial_k f \, \partial_i \partial_l g + \tfrac{1}{6} \partial_k \Pi^{ij} \partial_j \Pi^{kl} \partial_i f \, \partial_l g \big) + \mathcal{O}(\hbar^3).$$

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- Hyperbolic angle between vertices $p, q \in \mathbb{H}$:

$$\varphi(p, q) = \mathrm{Arg}\left(\frac{q-p}{q-\bar{p}}\right).$$

Weights

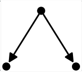
Definition

The *weight* of a Kontsevich graph $\Gamma \in G_n$ is given by the integral

$$w(\Gamma) = \frac{1}{(2\pi)^{2n}} \int_{C_n(\mathbb{H})} \bigwedge_{k=1}^{n} \mathrm{d}\varphi(p_k, p_{\mathrm{Left}(k)}) \wedge \mathrm{d}\varphi(p_k, p_{\mathrm{Right}(k)}),$$

over $C_n(\mathbb{H}) = \{(p_1, \ldots, p_n) \in \mathbb{H}^n : p_i \text{ pairwise distinct}\}$.

For example,



$$w\left(\cdot \right) = \int_{C_1(\mathbb{H})} \mathrm{d}\varphi(p, 0) \wedge \mathrm{d}\varphi(p, 1)$$
$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dx \int_0^{\infty} dy \frac{4y}{((x-1)^2 + y^2)(x^2 + y^2)} = \tfrac{1}{2}.$$

Formal series of graphs

Our approach: formal series of graphs, modulo ...

- For a fixed Poisson structure we have a correspondence

$$\Gamma \mapsto B_\Gamma.$$

- Can extend it to formal sums of graphs:

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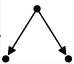
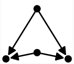
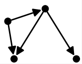
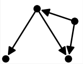
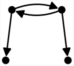
- **Main idea:** We consider formal sums modulo these things.

Recall:

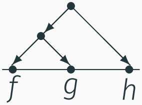
$\bullet \star \bullet = \bullet\bullet + \hbar$  $+ \hbar^2 \Big(\frac{1}{2}$  $+ \frac{1}{3}$  $+ \frac{1}{3}$  $+ \frac{1}{6}$  $\Big)$

$+ \mathcal{O}(\hbar^3).$

- Expand associator $(f \star g) \star h - f \star (g \star h)$ in terms of graphs:

$$\sum_{n=0}^{\infty} \hbar^n \sum_{r+s=n} B_r(B_s(f,g),h) - B_r(f, B_s(g,h)).$$

  Graphs act on graphs by the (iterated) Leibniz rule.
- Collect terms by using the skew-symmetry.
- What remains at $\hbar^2$: $\frac{2}{3}$ multiplied by



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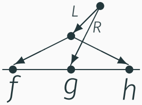

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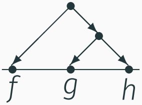
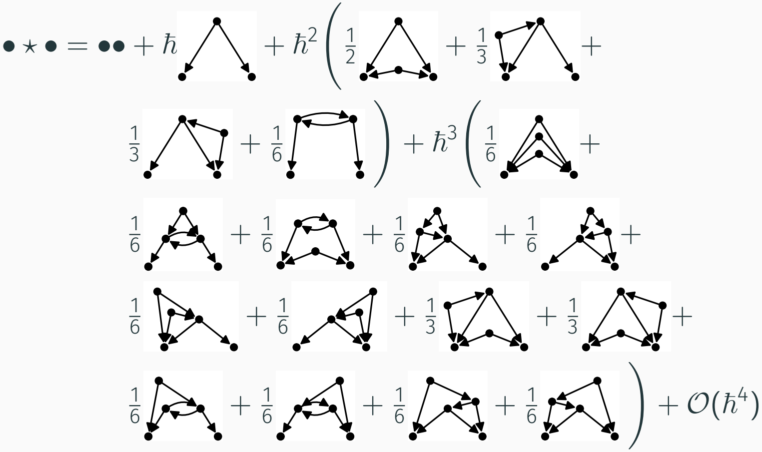


$= \mathrm{Jac}_{\Pi}(f,g,h) = 0.$

Illustration 2: Associativity up to order 3

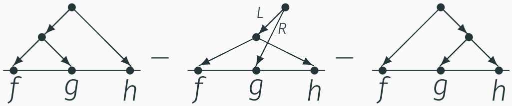


Expand $(f \star g) \star h - f \star (g \star h)$ graphically $\rightsquigarrow$ 39 terms at $\hbar^3$.
Vanishing is (differential) consequence of Jacobi, but how?

Jacobi identity and its differential consequences

We consider the Jacobiator as a tri-differential operator, we look at the differential consequences of its vanishing, and restrict to fixed total differential orders.

$\mathrm{Jac}_{\Pi}(f, g, h) =$



$= 0,$

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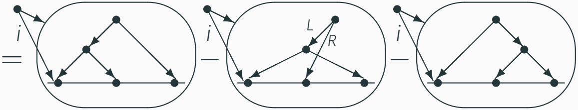
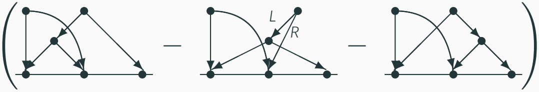


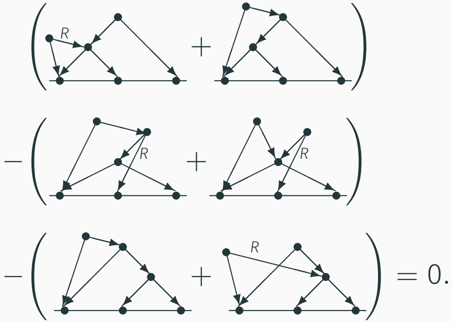
$= 0,$

$\frac{2}{3}\Pi^{ij}\,\mathrm{Jac}_{\Pi}(\partial_i f, \partial_j g, h) = \frac{2}{3}$

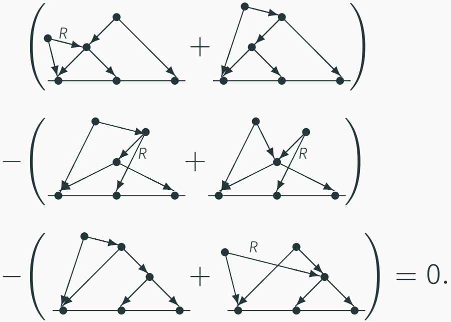


$= 0.$

Jacobi identity and its differential consequences, continued



Jacobi identity and its differential consequences, continued

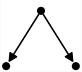
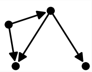
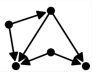


Moral: looking at formal sums of graphs is useful, since we can identify/construct differential consequences of the Jacobi identity graphically.

Illustration 3: undetermined weights & finding relations

For $n \geq 3$ direct integration becomes hard.
Start with undetermined weights, then find relations:

- skew-symmetry of graphs and weights,
- weight of mirror reflection of $\Gamma \in G_n$ is $(-1)^n w(\Gamma)$,
- for some graphs the weight integrand vanishes,
- multiplicativity of the weight; factorization into primes:

 $\times$  $=$  ,

- cyclic weight relations ({Willwacher, Shoikhet}–Felder),
- formula is universal, so holds for *all* Poisson bi-vectors: substitute Π into $\star$ with undetermined weights & solve associativity equation.

Primitive set of graphs

Definition (Primitive set of graphs)

A set of graphs is called *primitive* (or *basic*) if it contains only

- prime graphs
- of positive differential order,

and only one representative per equivalence class modulo {skew-symmetry, mirror reflections} is contained in the set.

Weights of all graphs are expressed via weights of basic graphs.

Strategy:

- generate basic set of graphs,
- find relations between their weights.

Substitution

The star product formula is *universal*, so it is associative for every Poisson structure.

- Evaluate associator at particular point for particular Π.
- For Π depending on a set of arbitrary functions $\{F_i\}$, consider the associator as a differential operator on f, g, h and the F_i.

This yields relations between the weights.

Remark: associator's $(1,2,3) \leftarrow (3,3),(1,5)$ have no primes at $\hbar^4 \implies$ relations between weights at order 3.

Result

At order 3 we have 15 basic graphs.

- Integrands vanishing: 1.
- Cyclic weight relations up to order 3.
- Substitute generic 3d Poisson, in associator up to order 4.

$\rightsquigarrow$ know all weights exactly, without calculating any integrals!

Result

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- Integrands vanishing: 1.
- Cyclic weight relations up to order 3.
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$\rightsquigarrow$ know all weights exactly, without calculating any integrals!

At order 4 we have 149 basic graphs.

- Integrands vanishing: 21.
- Cyclic weight relations up to order 6.
- Relations from associativity up to order 5 (in progress).

We *solve* the linear system of relations. This yields:

- 67 are now known exactly,
- 82 expressed via ≤ 10.

Conclusion: summary of results

Universal w.r.t. all Poisson structures:

- We understand factorization of associativity! (Lemma)
- The $\star$-product mod $\mathcal{O}(\hbar^4)$ known exactly.
- New linear relations between weights $w(\Gamma \in G_4)$ from $\star$ mod $\mathcal{O}(\hbar^6)$ $\rightsquigarrow$ in progress.
- 149 weights: 67 exactly, 82 via only ≤ 10 (in progress).
- Strategy to find weights fully implemented in software.
- Factorization algorithm applied to solve another problem (on tetrahedral flows).

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