QUANTIZATION AND NONCOMMUTATIVE GEOMETRY

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Symposium on advances in semi-classical methods in mathematics and physics

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In honour of H.J. Groenewold

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GROENEWOLD (1946)

- Main sources for Groenewold: von Neumann (1927, 1932) & Weyl (1927, 1931)
- "Our problems are about **α**: the correspondence between physical quantities and quantum operators (*quantization*) **β**: the possibility of understanding the statistical character of QM by averaging over uniquely determined processes as in classical statistical mechanics (*interpretation*)" [**α:** Weyl**, β:** von Neumann]
- One of the very few papers in the literature that *relates* **α** and **β:** Quantization, hidden variables, measurement, entanglement, EPR
- Impossibility of quantization respecting algebraic structure of classical mechanics (Groenewold-van Hove Theorem ☞ Geometric Quantization)
- "Star-product" (deformation of classical pointwise multiplication, which is recovered in limit $\hbar \to 0$) and Wigner function from Weyl's quantization rule

WEYL ON QUANTIZATION

- Weyl (1927) distinguished two very similar questions in QM:
  1. How to construct (i.e. mathematically) the self-adjoint operators corresponding to physical observables ("left open by von Neumann")
  2. What is the physical significance of these operators? ("solved by vN")

  Group theory answers 1. (*defining* a theory, cf. Wigner: *simplifying* a theory)
- Weyl (re)interpreted canonical commutation relations $[p,q] = -i\hbar$ as

  projective unitary representation of $\mathbb{R}^2$ (or representation of Heis group)

  p ☛ unitary representation U of $\mathbb{R}$: $U(a) = \exp(iap/\hbar)$

  q ☛ unitary representation V of $\mathbb{R}$: $V(b) = \exp(ivq)$

  } $[U(a),V(b)] \neq 0$
- Weyl's quantization formula for phase space functions f(p,q) added in 1931

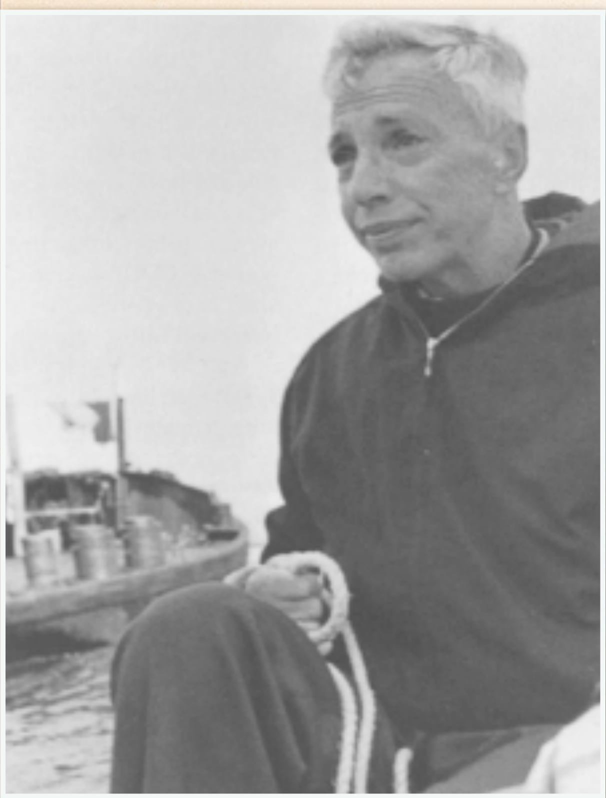
SYNOPSIS

- **Weyl's** views on quantization (1927-28) *bifurcated*:
  1. **Groenewold's** paper (1946): downplaying symmetry, emphasizing *deformation* and *classical limit,*
  2. **Mackey's** theory of quantization (1968) as *induced group representations* ("Weyl's Program"), *vice versa*

- **Rieffel's** *strict deformation quantization* (1989) realized by Lie groupoid C*-algebras (1998) provides synthesis ("Weyl - Groenewold Program")

Hermann Weyl
(1885-1955)

George Mackey
(1916-2006)



"Hip" Groenewold
(1910-1996)

Marc Rieffel
(1937)

MACKEY'S QUANTIZATION

- Mackey *breaks phase space symmetry between p and q*; regards CCR $[p,q] = -i\hbar$ as infinitesimal *system of imprimitivity*:

  p ☛ 1) *unitary representation* of $\mathbb{R}$: $U(a) = \exp(iap/\hbar)$

  q ☛ 2) *projection-valued measure* $E \longmapsto P(E)$ on $\mathbb{R}$

  CCR ☛ 3) *Covariance condition* $U(a)P(E)U(a)^* = P(a \cdot E)$

- Generalization: group G acts on (configuration) space M

  1) *unitary representation* of G on Hilbert space $\mathcal{H}$

  2) *projection-valued measure* (PVM) on M (on same $\mathcal{H}$)

  3) *Covariance condition* $U(g)P(E)U(g)^* = P(g \cdot E)$

INDUCED REPRESENTATIONS

- Quantization à la Mackey = representation theory of systems of imprimitivity, with important special case:
- If G acts transitively, so $M \cong G/H$, then there is a natural bijective correspondence (preserving unitary equivalence) between systems of imprimitivity and continuous unitary representations of G induced from unitary reps of H

Recovers uniqueness of the CCR: $G = M = \mathbb{R}^3$, $H = \{e\}$

"Explains" spin: $G = E(3) = SO(3) \ltimes \mathbb{R}^3$, $M = \mathbb{R}^3$, $H = SO(3)$

COMMUTATION RELATIONS

- G *Lie group*, M *manifold*, system of imprimitivity i.e. unitary rep U(G) on Hilbert space $\mathcal{H}$ plus G-covariant projection-valued measure $E \longmapsto P(E)$ on M in $\mathcal{H}$, then, for X, Y in Lie (G) and C^∞ functions f, g on M, obtain CCR for "momentum" and "position" operators:

$$Q_\hbar(X) = i\hbar dU(X) \quad \text{and} \quad Q_\hbar(f) = \int_M dE(x)\, f(x)$$

$$[Q_\hbar(X), Q_\hbar(Y)] = i\hbar\, Q_\hbar([X,Y]), \quad [Q_\hbar(f), Q_\hbar(g)] = 0,$$

$$[Q_\hbar(X), Q_\hbar(f)] = i\hbar\, Q_\hbar(X_M f), \quad (X_M \text{ vector field on M})$$

GROUPOID C*-ALGEBRAS

- **Groupoid** = small category where each arrow is invertible ≈ "group" with *partial* multiplication (but inverse defined everywhere), e.g. *space* $\Gamma = M$, or:

1) *pair groupoid* $\Gamma = M \times M$, product $(x,y) \cdot (x', y')$ defined iff $x' = y$, result (x, y')

2) *semi-direct product groupoid* $\Gamma = G \rtimes M$: given G-action on M, as a set $\Gamma = G \times M$ with product $(g, x) \cdot (h, y)$ defined iff $y = g^{-1} x$, resulting in (gh, x)

- **C*-algebra** = "nice" algebra of bounded operators on Hilbert space, also defined abstractly, with ensuing representation theory on Hilbert spaces, like groups (Groningen C*mathematical physics school: Hugenholtz, Winnink)
- **Lie groupoid** *canonically* G defines C*-algebra C*(G) (Connes, 1980s)
- Systems of imprimitivity for given G-action on M (and hence Mackey's quantization) bijectively correspond to representations of corresponding semi-direct product groupoid C*-algebra $C^*(G \rtimes M)$ $(\cong G \rtimes C_0(M))$

INTERIM SCORE

- Mackey's development of Weyl's quantization program (perhaps reformulated in terms of semi-direct Lie groupoid C*-algebras) is entirely based on symmetry and operator theory
- Groenewold's development of Weyl's program in terms of deformation & classical limit (downplaying symmetry) seems lost
- Best of both worlds is possible: Lie *groupoid* C*-algebras are "deformations" of Lie *algebroid* Poisson algebras/manifolds
- This also justifies the (heavy) groupoid & C*-algebra language!
- Settings for deformation quantization: *formal* (= purely algebraic) (Berezin, Flato, ..., Kontsevich) or *strict* (= C*-algebraic) (Rieffel)

STRICT DEFORMATION QUANTIZATION

- ***Poisson manifold*** (Weinstein, 1983) is manifold with Lie bracket on (commutative) algebra of smooth functions satisfying a Leibniz rule
- ***Continuous field of C*-algebras*** (Dixmier, 1962) is (not necessarily locally trivial!!!!!!!!!!) fiber bundle whose fibers are "glued" C*-algebras
- ***Strict quantization*** of Poisson manifold P is continuous field of C*-algebras over $I \subseteq [0, 1]$ with *commutative* C*-algebra $A_0 = C_0(P)$ and *non-commutative* C*-algebras at $\hbar > 0$, plus *quantization maps*

  $Q_\hbar : A_0 \to A_\hbar$ satisfying the *Dirac - Groenewold - Rieffel condition*

$$\lim_{\hbar \to 0} \left\| \frac{i}{\hbar} [Q_\hbar(f), Q_\hbar(g)] - Q_\hbar(\{f, g\}) \right\|_{A_\hbar} = 0$$

LIE ALGEBROIDS

- **Lie algebroid** over manifold M is vector bundle p: E→M equipped with *second* projection a: E→TM (anchor map) *and* Lie bracket on smooth sections s of E→M such that [s, f · s'] = f · [s,s'] + a(s)f · s'
- Each Lie groupoid Γ (which is a special category over some object/base space M) defines a Lie algebroid Lie(Γ) over the same base space
- Examples: *Lie group* ☛Lie algebra, *smooth pair groupoid* M x M ☛ TM, *semi-direct product* Γ = G ⋊ M ☛ Lie(G) ⋊ M (Lie bracket: G-action)
- Key point: *dual bundle E* to Lie algebroid E is canonically a Poisson manifold* (Courant, Weinstein, 1990), generalizing *Lie-Poisson bracket* on Lie(G)*: $\{\underline{X},\underline{Y}\} = \underline{[X, Y]}$ where X∈ Lie(G) defines (linear and hence smooth) function $\underline{X}$ on Lie(G)* by X(θ) = θ(X), where θ ∈ Lie(G)*

MAIN THEOREM

- Lie groupoid Γ $\left\{\right.$
  - ☛ C*-algebra $C^*(\Gamma)$
  - ☛ Lie algebroid $\mathrm{Lie}(\Gamma)$
- Lie algebroid $\mathrm{Lie}(\Gamma)$ ☛ Poisson manifold $\mathrm{Lie}(\Gamma)^*$
- Fibers $A_0 = C_0(\mathrm{Lie}(\Gamma)^*)$ at $\hbar = 0$ and $A_\hbar = C^*(\Gamma)$ at all $\hbar > 0$ (trivially glued) form a continuous field of C*-algebras
- Generalized Weyl quantization map satisfies the *Dirac - Groenewold - Rieffel* condition (relating commutator to PB)

$C^*(\Gamma)$ is a strict deformation quantization of $\mathrm{Lie}(\Gamma)^*$

A WORD ABOUT THE PROOF

- The continuous cross-sections of a continuous field of C*-algebras ($A_\hbar$) form a C*-algebra A (from which fibers $A_\hbar$ and continuity structure can be recovered)
- In our case A is itself a groupoid C*-algebra (viz. of Connes's *tangent groupoid* to the given Lie groupoid Γ)
- This also provides a technique to prove Atiyah-Singer type index theorems (so these are related to quantization!)

SUMMARY

- Weyl's program of constructing the operators of quantum mechanics from symmetry arguments can be carried out in the spirit of Groenewold's emphasis on deformation and classical limit

- Result: *$C^*(\Gamma)$ is a strict deformation quantization of Lie$(\Gamma)^*$*
  Generalizes Mackey's quantization of group actions by systems of imprimitivity (which comes out as the special case $\Gamma = G \rtimes M$)

- Weyl's Problem 2: "*What is the physical significance of these operators?*"
  Groenewold's Problem β: *"understanding the statistical character of QM by averaging as in classical statistical mechanics (interpretation)"* remain!

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