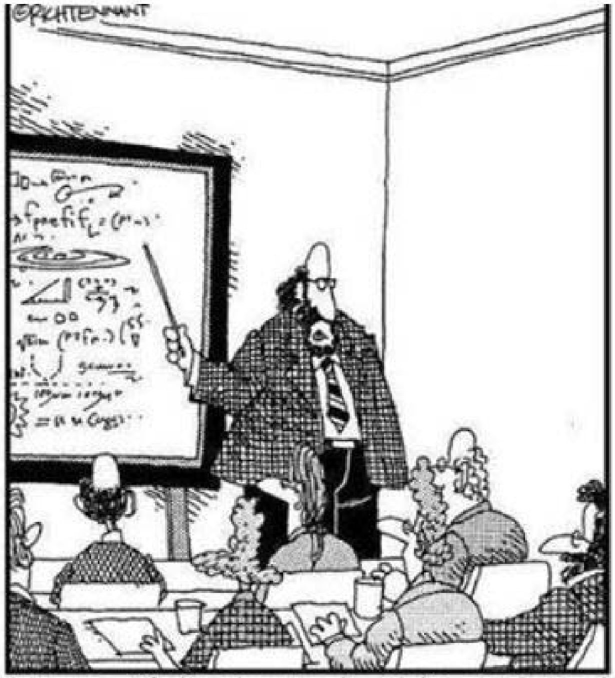
Semiclassical approximation,

Nonlinear eigenvalue problems,

and *PT*-symmetric quantum mechanics



"Along with 'Antimatter,' and 'Dark Matter,' we've recently discovered the existence of 'Doesn't Matter,' which appears to have no effect on the universe whatsoever."

Carl M. Bender
Washington University

Groningen
October 2016

The idea of *PT*-symmetric quantum theory:

Replace the mathematical condition of Hermiticity by the *weaker* and *physical* condition of *PT* symmetry, where

P *= parity,* ***T*** *= time reversal*

***Physical* because *P* and *T* are elements of the Lorentz group.**

Two examples: *cubic* and *quartic* potentials

(1) $$H = p^2 + ix^3$$

These Hamiltonians have ***PT*** **symmetry!**

(2) $$H = p^2 - x^4$$

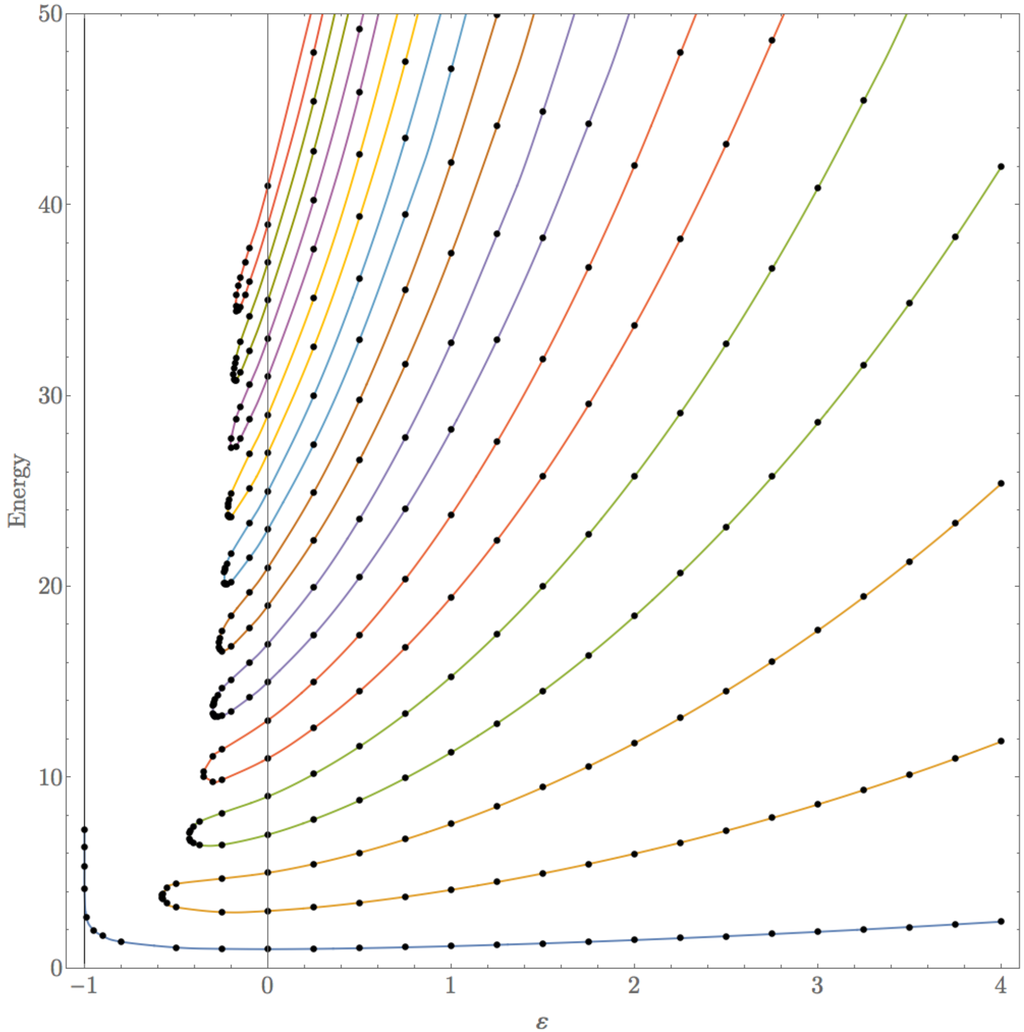
← ***An upside-down potential with real positive eigenvalues!***

Z. Ahmed, CMB, and M. V. Berry, *J. Phys. A: Math. Gen.* **38**, L627 (2005) [arXiv: quant-ph/0508117]

CMB, D. C. Brody, J.-H. Chen, H. F. Jones, K. A. Milton, and M. C. Ogilvie, *Phys. Rev. D* **74**, 025016 (2006) [arXiv: hep-th/0605066]

A class of *PT*-symmetric Hamiltonians:

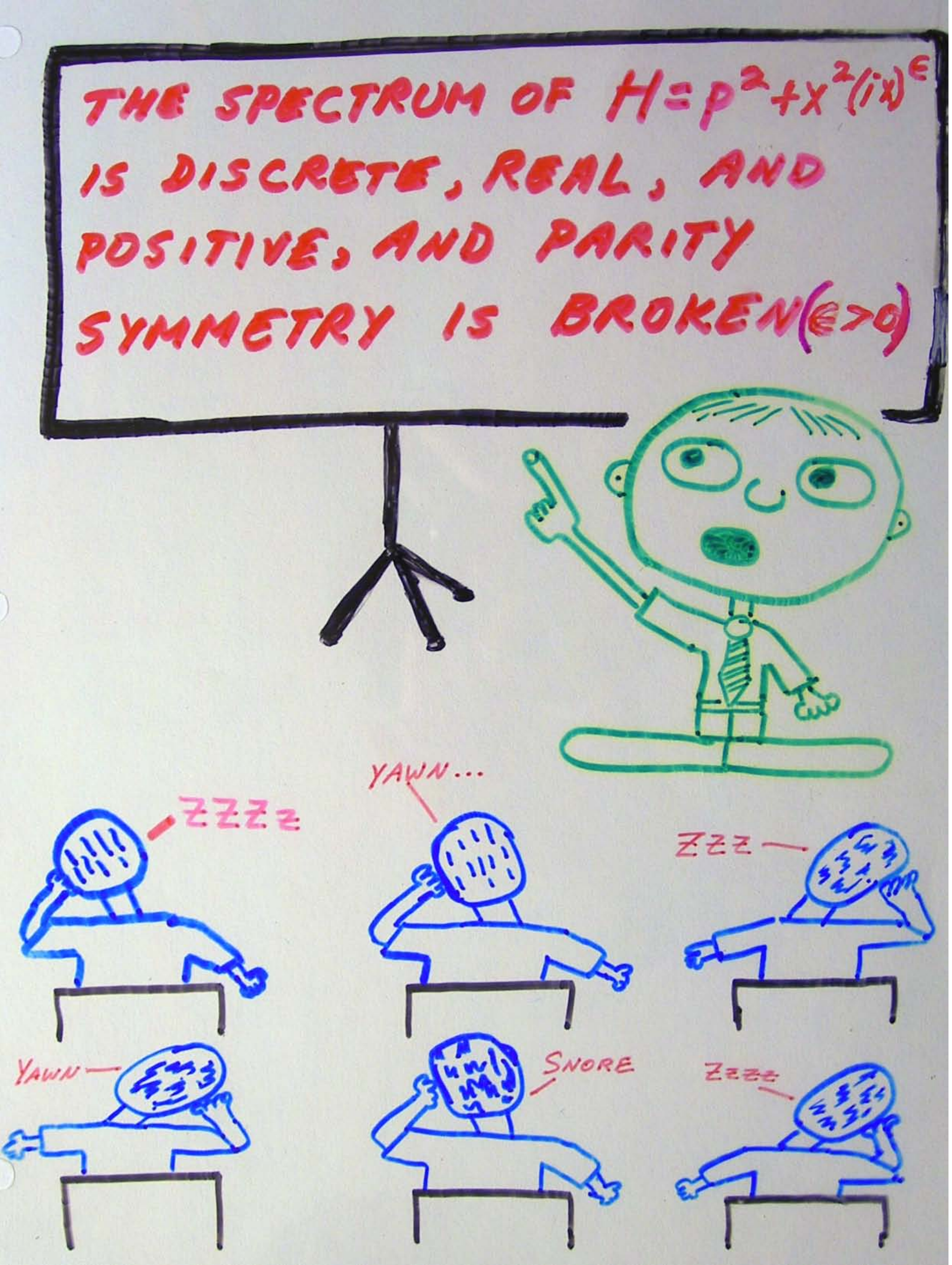
$$H = p^2 + x^2(ix)^\varepsilon \qquad (\varepsilon \text{ real})$$



Look! The energies are real, positive, and discrete for ε > 0 (!!)

cubic: $\varepsilon = 1$
quartic: $\varepsilon = 2$

CMB and S. Boettcher
Physical Review Letters **80**, 5243 (1998)

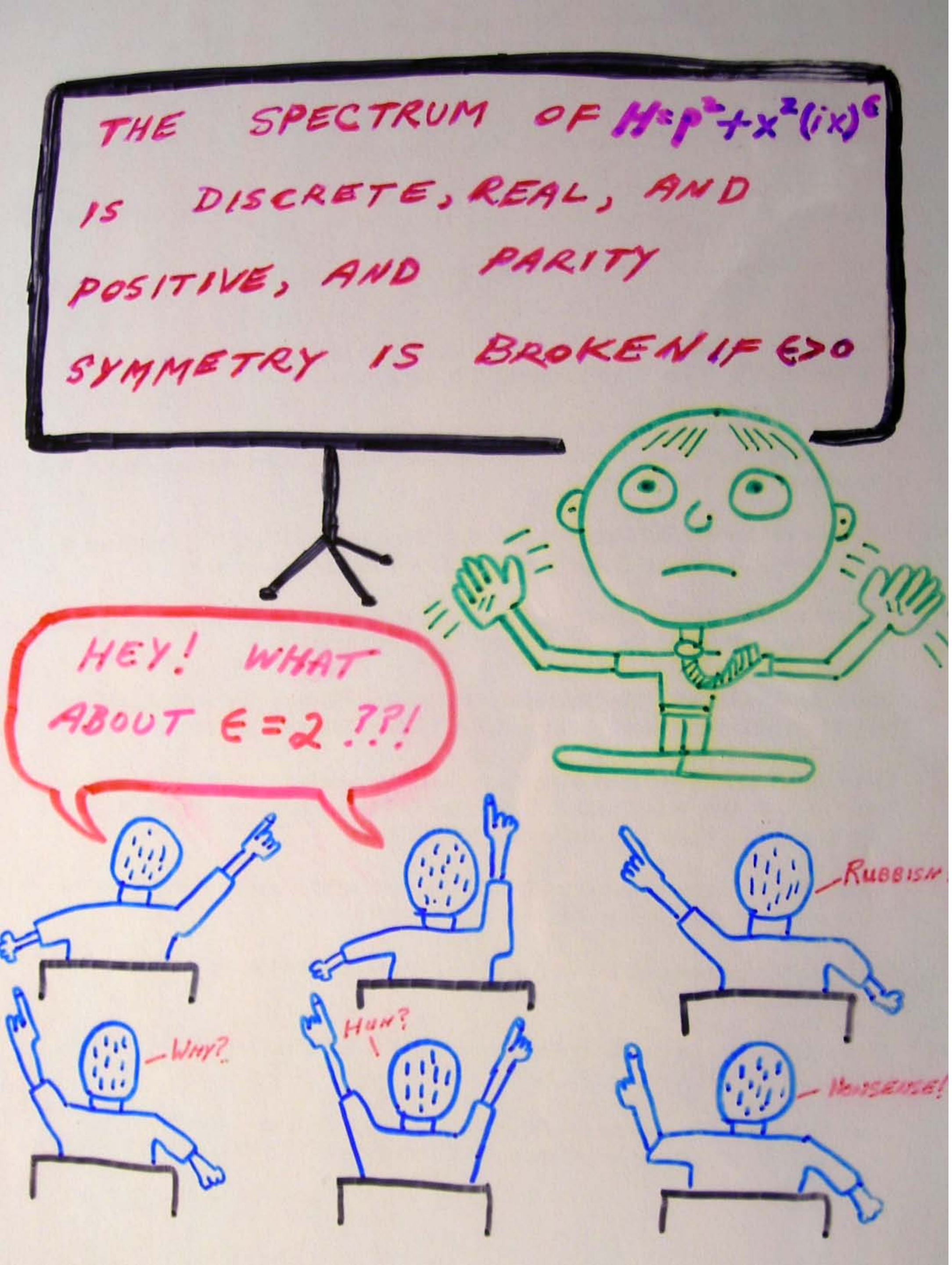


Rigorous proof of real eigenvalues:

"ODE/IM Correspondence"
P. Dorey, C. Dunning, and R. Tateo,
J. Phys. A **40**, R205 (2007)

PT **symmetry *controls instabilities***

Physical systems that you might *think* are unstable become *stable* in the complex domain…



Upside-down potential with real positive eigenvalues?!

$$V(x) = -x^4$$

Z. Ahmed, CMB, and M. V. Berry,
J. Phys. A: Math. Gen. **38**, L627 (2005)
[arXiv: quant-ph/0508117]

CMB, D. C. Brody, J.-H. Chen, H. F. Jones, K. A. Milton, and M. C. Ogilvie,
Phys. Rev. D **74**, 025016 (2006)
[arXiv: hep-th/0605066]

Donald Trump believes in *PT*…

***P* and *T* fit together nicely…**

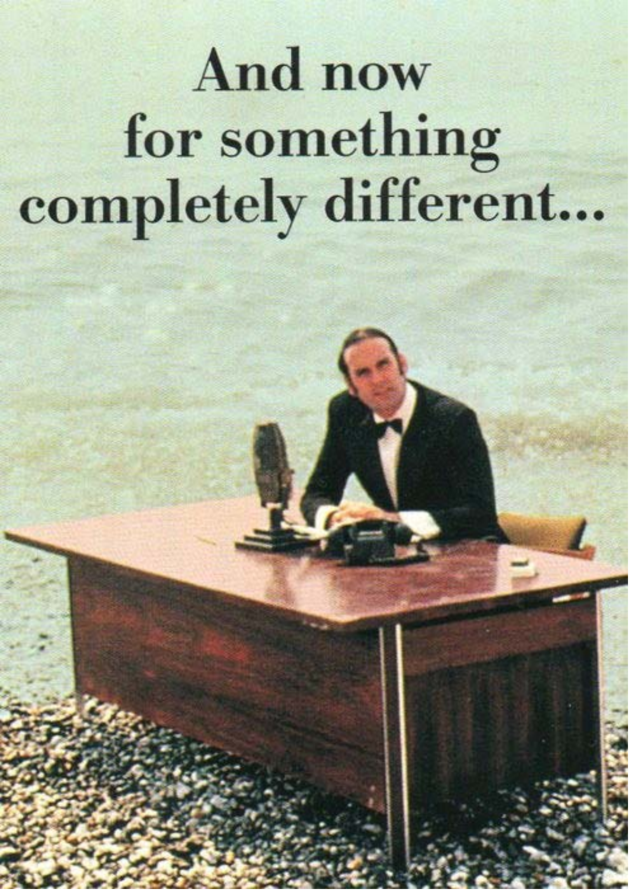
Stability of the Higgs vacuum:

"***PT***-symmetric interpretation of unstable effective potentials"
CMB, D. W. Hook, N. E. Mavromatos, and S. Sarkar
Journal of Physics A **49**, 45LT01 (2016) [arXiv: 1506.01970]

Stability of the double-scaling limit in QM and QFT:

"***PT***-symmetric Interpretation of double-scaling"
CMB, M. Moshe, and S. Sarkar
Journal of Physics A **46**, 102002 (2013) [arXiv: 1206.4943]

"Double-scaling limit of the O(N)-symmetric anharmonic oscillator"
CMB and S. Sarkar
Journal of Physics A **46**, 442001 (2013) [arXiv: 1307.4348]



Instabilities associated with nonlinear eigenvalue problems...

CMB, A. Fring, Q. Wang, and J. Komijani

Linear eigenvalue problems...

$$-\psi''(x) + V(x)\psi(x) = E\psi(x) \qquad \psi(\pm\infty) = 0$$

For linear problems *WKB* gives a good approximation for large eigenvalues

$$\int_{x_1}^{x_2} dx \sqrt{E_n - V(x)} \sim (n + 1/2)\pi \quad (n \to \infty)$$

*n*th energy level grows like a constant times a power of *n*

Example 1: harmonic oscillator

$$V(x) = x^2$$

$$E_n \sim n \quad (n \to \infty)$$

Example 2: anharmonic oscillator

$$V(x) = x^4$$

$$E_n \sim B n^{4/3} \quad (n \to \infty)$$

$$B = \left[\frac{3\Gamma(3/4)\sqrt{\pi}}{\Gamma(1/4)}\right]^{4/3}$$

WKB works for _PT_-symmetric Hamiltonians as well:

$$H = p^2 + x^2(ix)^\varepsilon \qquad (\varepsilon \text{ real})$$

$$E_n \sim \left[\frac{\Gamma\left(\frac{3}{2}+\frac{1}{\varepsilon+2}\right)\sqrt{\pi}\,n}{\sin\left(\frac{\pi}{\varepsilon+2}\right)\Gamma\left(1+\frac{1}{\varepsilon+2}\right)}\right]^{\frac{2\varepsilon+4}{\varepsilon+4}} \qquad (n \to \infty)$$

Hyperasymptotics

Leading asymptotic behavior of solutions to

$$-\psi''(x) + V(x)\psi(x) = E\psi(x)$$

for large positive *x*:

$$\psi(x) \sim C[V(x) - E]^{-1/4} \exp\left[\int^x ds\sqrt{V(s) - E}\right] \quad (x \to \infty)$$

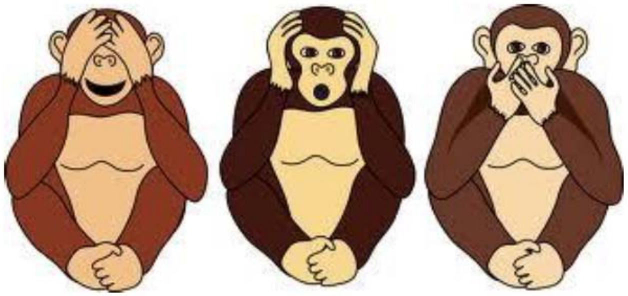
NOTE: Only *ONE* arbitrary constant.

Second arbitrary constant invisible because it is contained in the *subdominant* solution:

$$\psi(x) \sim D[V(x) - E]^{-1/4} \exp\left[-\int^x ds\sqrt{V(s) - E}\right] \quad (x \to \infty)$$

***Physical* solution is *Unstable* under small changes in *E*.**

Three characteristic properties of solutions



(1) **Oscillatory** in *classically allowed* region (*n*th eigenfunction has *n* nodes)

(2) **Monotone decay** in *classically forbidden* region

(3) **Transition** at the boundary (*turning point*)

Nonlinear toy eigenvalue problem

$$y'(x) = \cos[\pi x y(x)], \quad y(0) = a$$

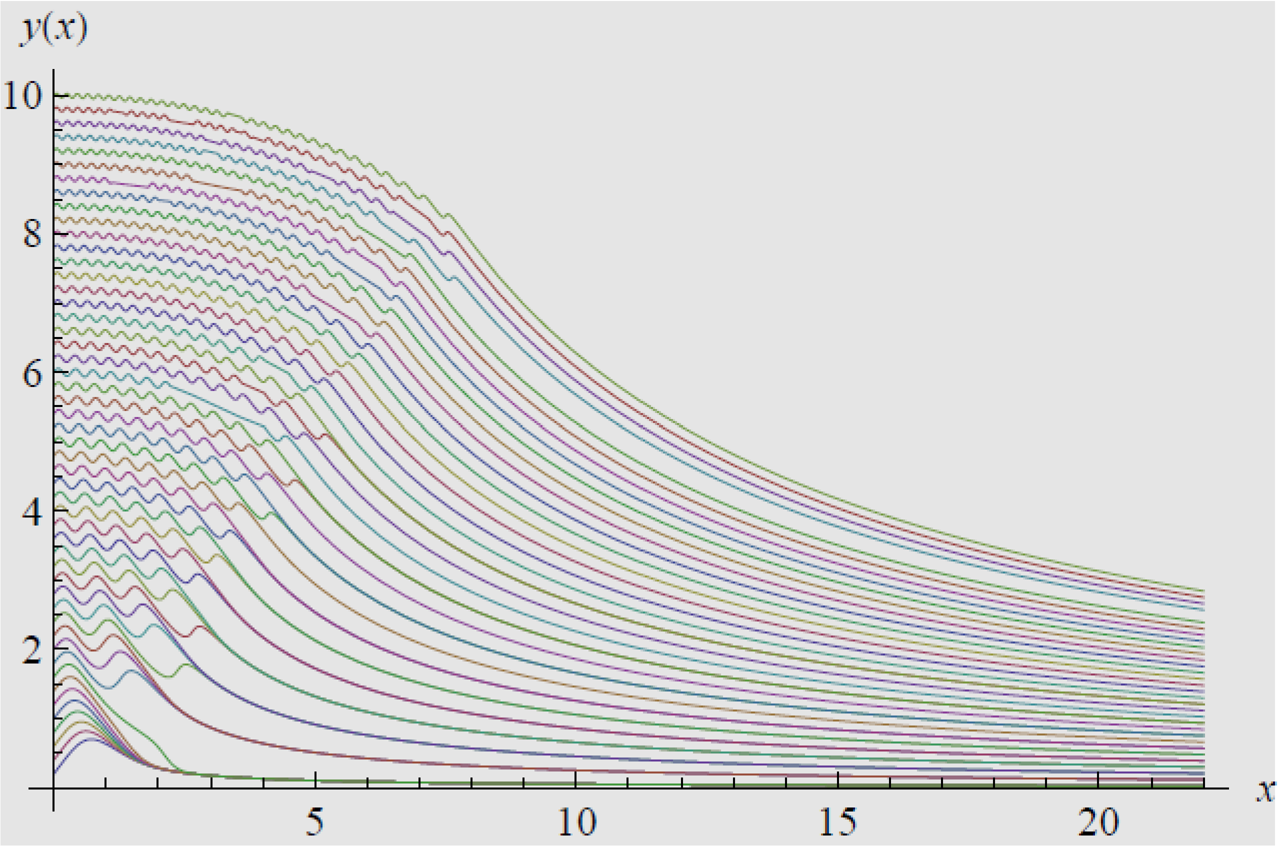
Some references:

[1] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw Hill, New York, 1978), chap. 4.

[2] C. M. Bender, D. W. Hook, P. N. Meisinger, and Q. Wang, Phys. Rev. Lett. **104**, 061601 (2010).

[3] C. M. Bender, D. W. Hook, P. N. Meisinger, and Q. Wang, Ann. Phys. **325**, 2332-2362 (2010).

[4] J. Gair, N. Yunes, and C. M. Bender, J. Math. Phys. **53**, 032503 (2012).



Solutions for 50 initial conditions

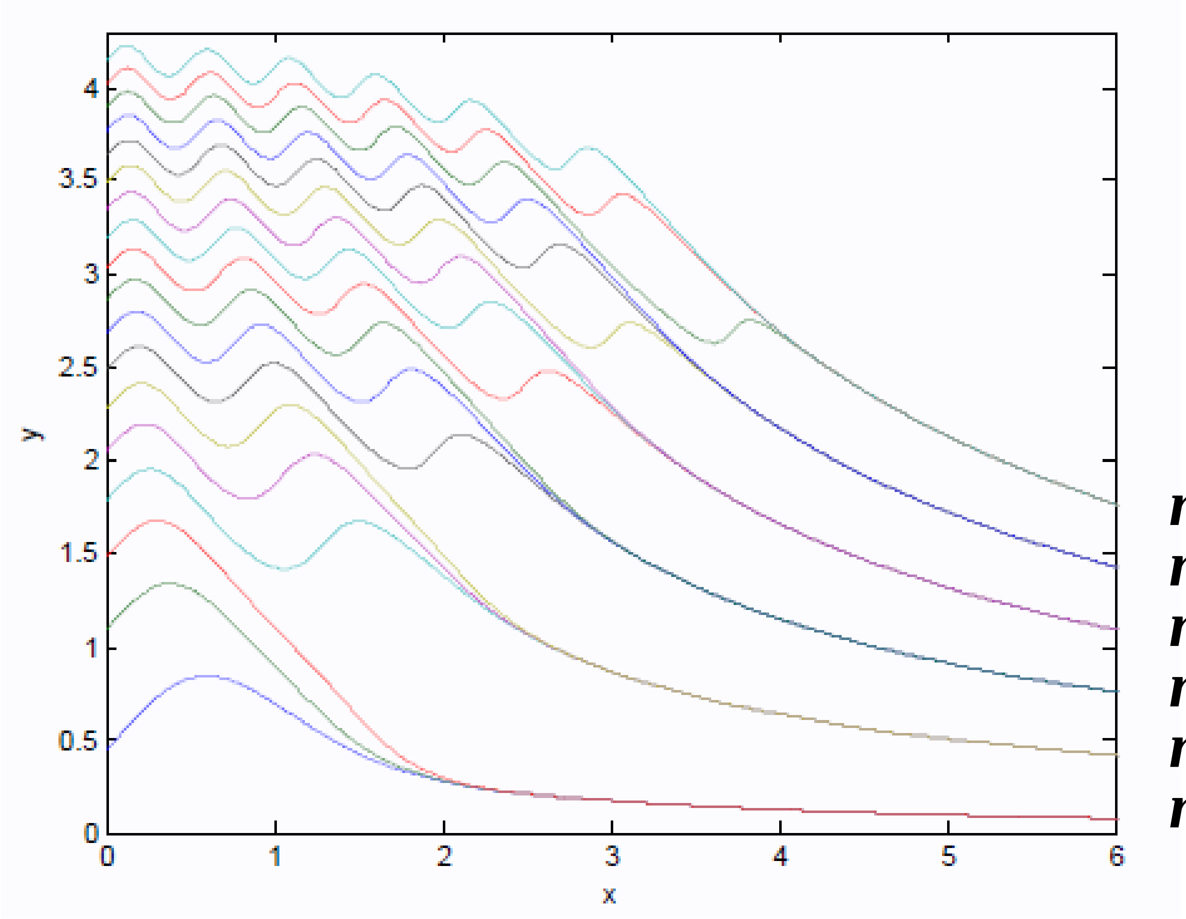
Note: (1) oscillation (2) monotone decay (3) transition

Asymptotic behavior for large *x*

Solution behaves like: $$y(x) \sim \frac{m+1/2}{x}$$

m **= 0, 1, 2, 3, ... is an integer**

There's a ***big*** problem here...



m **= 10**

m **= 8**

m **= 6**

m **= 4**

m **= 2**

m **= 0**

Where are the odd-*m* solutions?!?

Furthermore, no arbitrary constant appears in the asymptotic behavior!!

Where is the arbitrary constant?!?

Is it in higher order?

Higher-order asymptotic behavior for large *x* still contains no arbitrary constant!

$$y(x) \sim \frac{m+1/2}{x} + \sum_{k=1}^{\infty} \frac{c_k}{x^{2k+1}} \quad (x \to \infty)$$

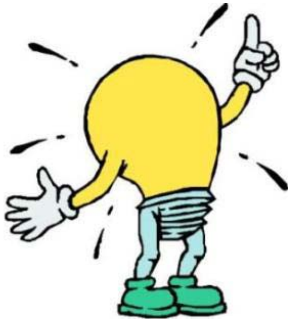
$$\begin{aligned}
c_1 &= \frac{(-1)^m}{\pi}(m+1/2), \\
c_2 &= \frac{3}{\pi^2}(m+1/2), \\
c_3 &= (-1)^m \left[\frac{(m+1/2)^3}{6\pi} + \frac{15(m+1/2)}{\pi^3}\right], \\
c_4 &= \frac{8(m+1/2)^3}{3\pi^2} + \frac{105(m+1/2)}{\pi^4}, \\
c_5 &= (-1)^m \left[\frac{3(m+1/2)^5}{40\pi} + \frac{36(m+1/2)^3}{\pi^3} + \frac{945(m+1/2)}{\pi^5}\right], \\
c_6 &= \frac{38(m+1/2)^5}{15\pi^2} + \frac{498(m+1/2)^3}{\pi^4} + \frac{10395(m+1/2)}{\pi^6}.
\end{aligned}$$

Hyperasymptotic analysis

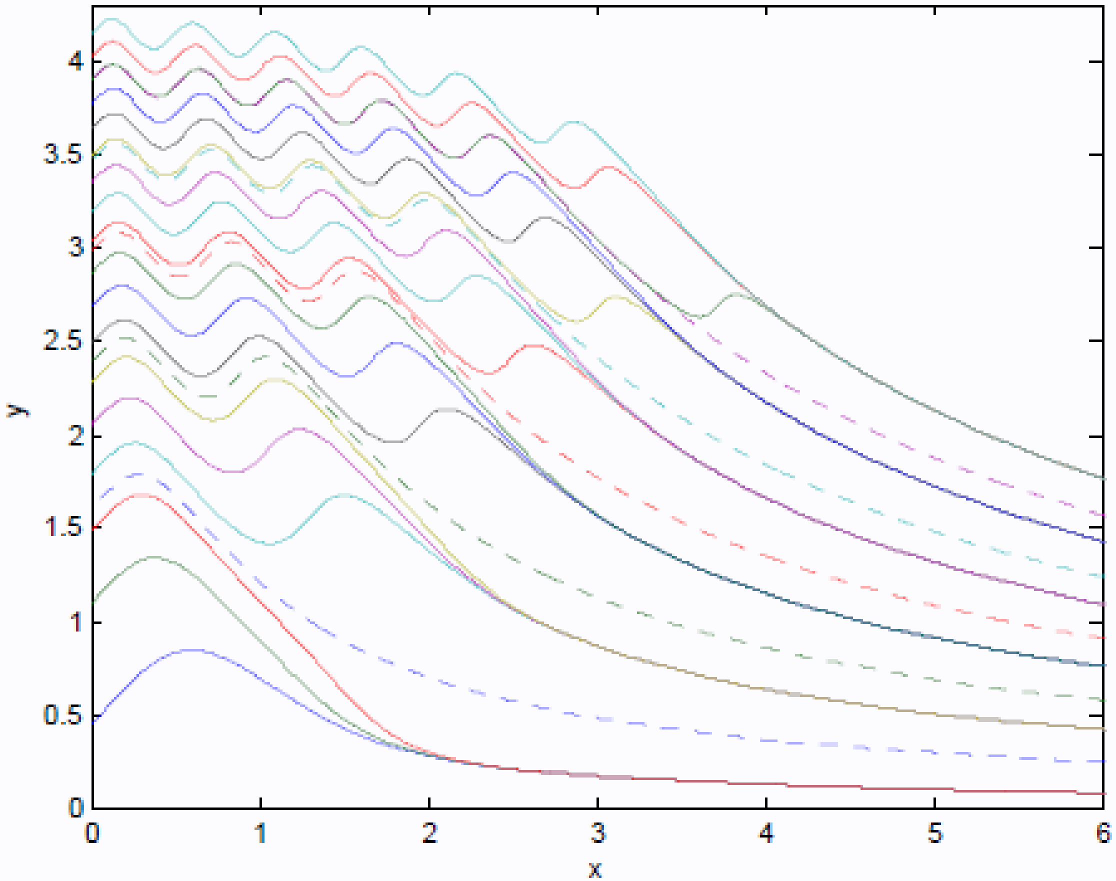
Difference of two solutions
in one bundle: $Y(x) \equiv y_1(x) - y_2(x)$

$$\begin{aligned} Y'(x) &= \cos[\pi x y_1(x)] - \cos[\pi x y_2(x)] \\ &= -2\sin\left[\tfrac{1}{2}\pi x y_1(x) + \tfrac{1}{2}\pi x y_2(x)\right]\sin\left[\tfrac{1}{2}\pi x y_1(x) - \tfrac{1}{2}\pi x y_2(x)\right] \\ &\sim -2\sin\left[\pi\left(m+\tfrac{1}{2}\right)\right]\sin\left[\tfrac{1}{2}\pi x Y(x)\right] \quad (x\to\infty) \\ &\sim -(-1)^m \pi x Y(x) \quad (x\to\infty). \end{aligned}$$

$$Y(x) \sim K\exp\left[-(-1)^m \pi x^2\right] \quad (x\to\infty)$$



Aha! *K* is the arbitrary constant!
Odd-*m* solutions are *unstable*,
even-*m* solutions are *stable*.



$m = 9$
$m = 7$
$m = 5$
$m = 3$
$m = 1$

$$y(0) = a \in \{1.6026, 2.3884, 2.9767, 3.4675, 3.8975, 4.2847, \ldots\}$$

Eigenvalues **correspond to odd-*m* initial values.**

Eigenfunctions **are (*unstable*) *separatrices*, which begin at eigenvalues.**

We calculated up to *m*=500,001

Let $m = 2n - 1$

For large *n* the *n*th eigenvalue grows like the *square root* of *n* times a constant *A*, and we used Richardson extrapolation to show that

A= 1.7817974363...

and then we guessed *A*.

Result:

$$a_n \sim A\sqrt{n} \quad (n \to \infty)$$

$$A = 2^{5/6}$$

This is a nontrivial problem...

Analytic calculation of the constant *A*

Construct moments of *z*(*t*):

$$A_{n,k}(t) \equiv \int_0^t ds \cos[n\lambda s z(s)] \frac{s^{k+1}}{[z(s)]^k}$$

Moments are associated with a semi-infinite linear one-dimensional random walk in which random walkers become static as they reach n=1

$$2\alpha_{1,k} + \alpha_{2,k-1} = 0, \qquad 2\alpha_{n,k} + \alpha_{n-1,k-1} + \alpha_{n+1,k-1} = 0 \quad (n \geq 3).$$

$$2\alpha_{2,k} + \alpha_{3,k-1} = 0,$$

Solve the random walk problem exactly and get $A = 2^{5/6}$

CMB, A. Fring, and J. Komijani
J. Phys. A: Math. Theor. **47**, 235204 (2014)
[arXiv: math-ph/1401.6161]

Three ***nontrivial*** second-order nonlinear eigenvalue problems

Painlevé equations

Paul Painlevé
(1863-1933)

Six Painlevé equations known as Painlevé I – VI

Only spontaneous singularities are poles

Painlevé I

$$\frac{d^2y}{dt^2} = 6y^2 + t$$

Painlevé II

$$\frac{d^2y}{dt^2} = 2y^3 + ty + \alpha$$

Painlevé III

$$ty\frac{d^2y}{dt^2} = t\left(\frac{dy}{dt}\right)^2 - y\frac{dy}{dt} + \delta t + \beta y + \alpha y^3 + \gamma t y^4$$

Painlevé IV

$$y\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{dy}{dt}\right)^2 + \beta + 2(t^2 - \alpha)y^2 + 4ty^3 + \frac{3}{2}y^4$$

Painlevé V

$$\frac{d^2y}{dt^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1}$$

Painlevé VI

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)$$

(1) First Painlevé transcendent

$$y''(t) = 6[y(t)]^2 + t, \qquad y(0) = b,\ y'(0) = c$$

Solution *y*(*x*) must *choose* between two possible asymptotic behaviors as *x* gets large and negative:

$$+\sqrt{-t/6} \text{ or } -\sqrt{-t/6}.$$

Example of a *difficult* choice ...

Two possible asymptotic behaviors

Lower square-root branch is ***stable***:

$$y(x) \sim -\sqrt{-x} + c(-x)^{-1/8}\cos\left[\tfrac{4}{5}\sqrt{2}(-x)^{5/4} + d\right] \quad (x \to -\infty)$$

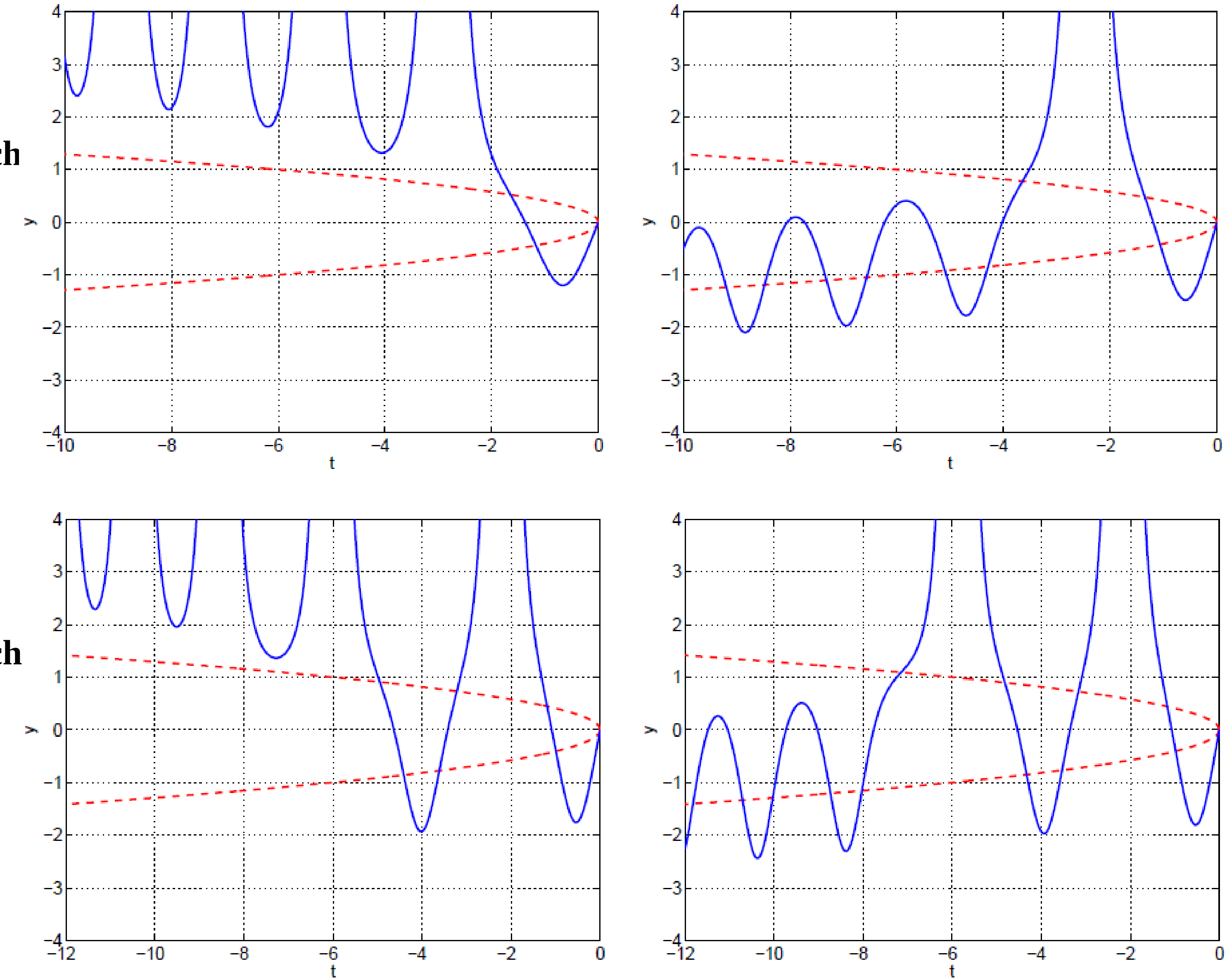
Upper square-root branch is ***unstable***:

$$y(x) \sim \sqrt{-x} + c_{\pm}(-x)^{-1/8}\exp\left[\pm\tfrac{4}{5}\sqrt{2}(-x)^{5/4}\right] \quad (x \to -\infty)$$

Two possible kinds of solutions (NOT eigenfunctions):

Unstable branch

Stable branch



Unstable branch

Stable branch

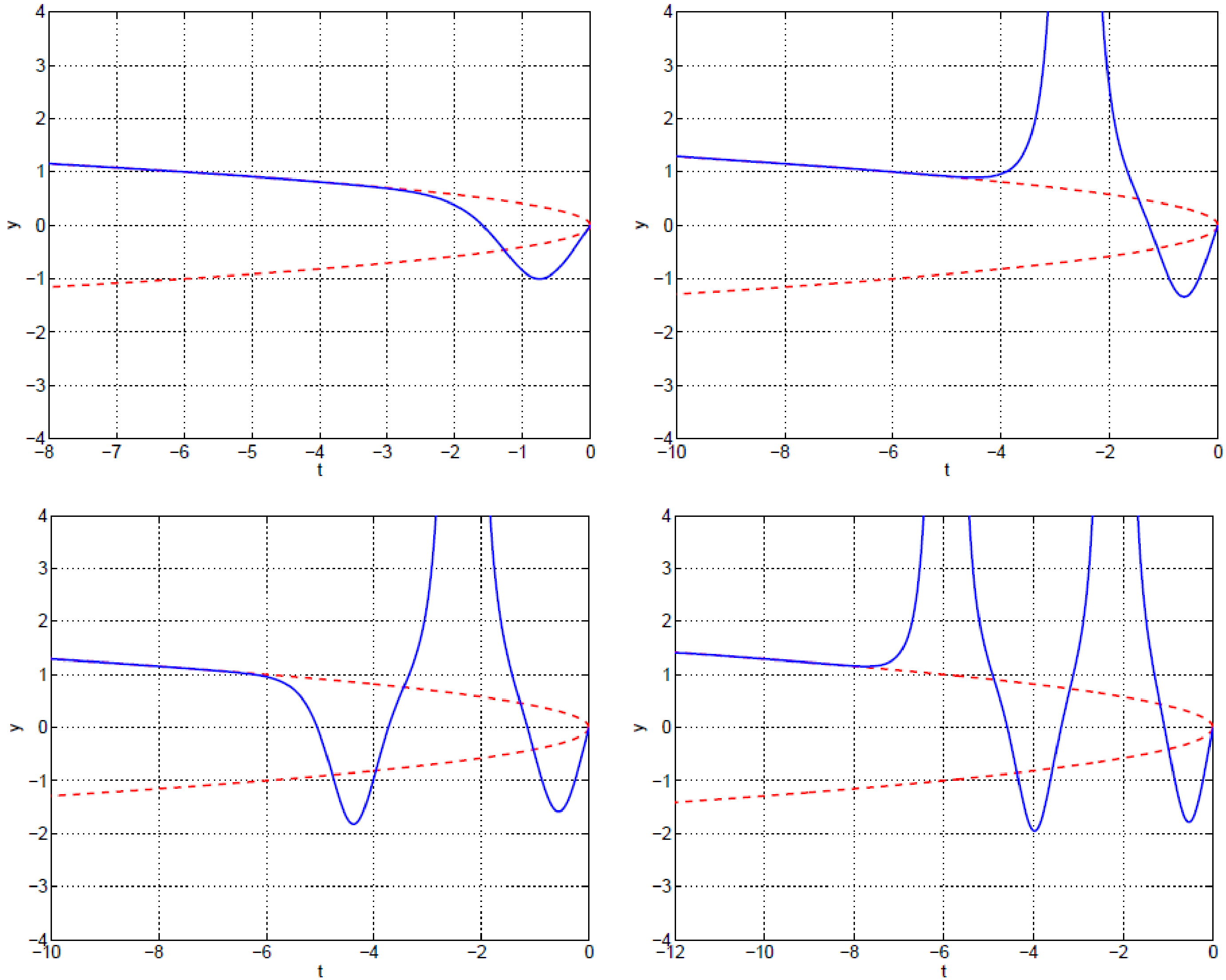
First four separatrix (eigenfunction) solutions:

Unstable

Stable

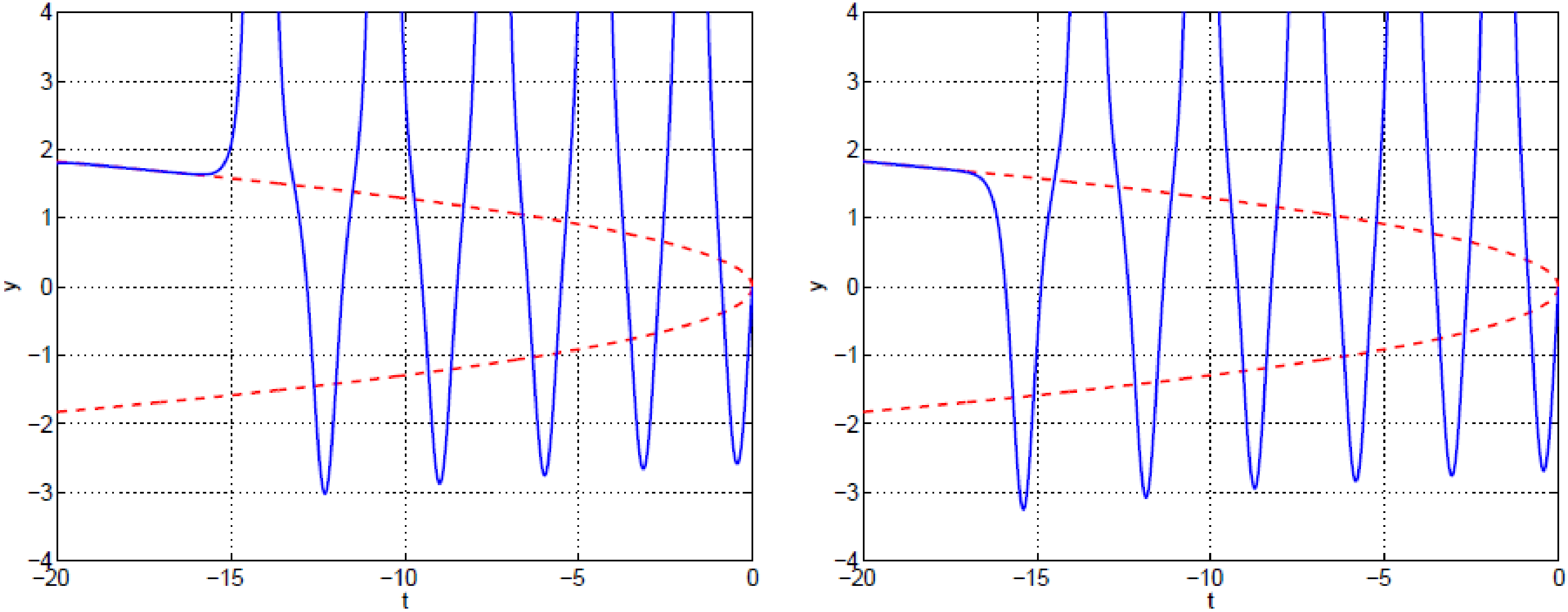
Unstable

Stable



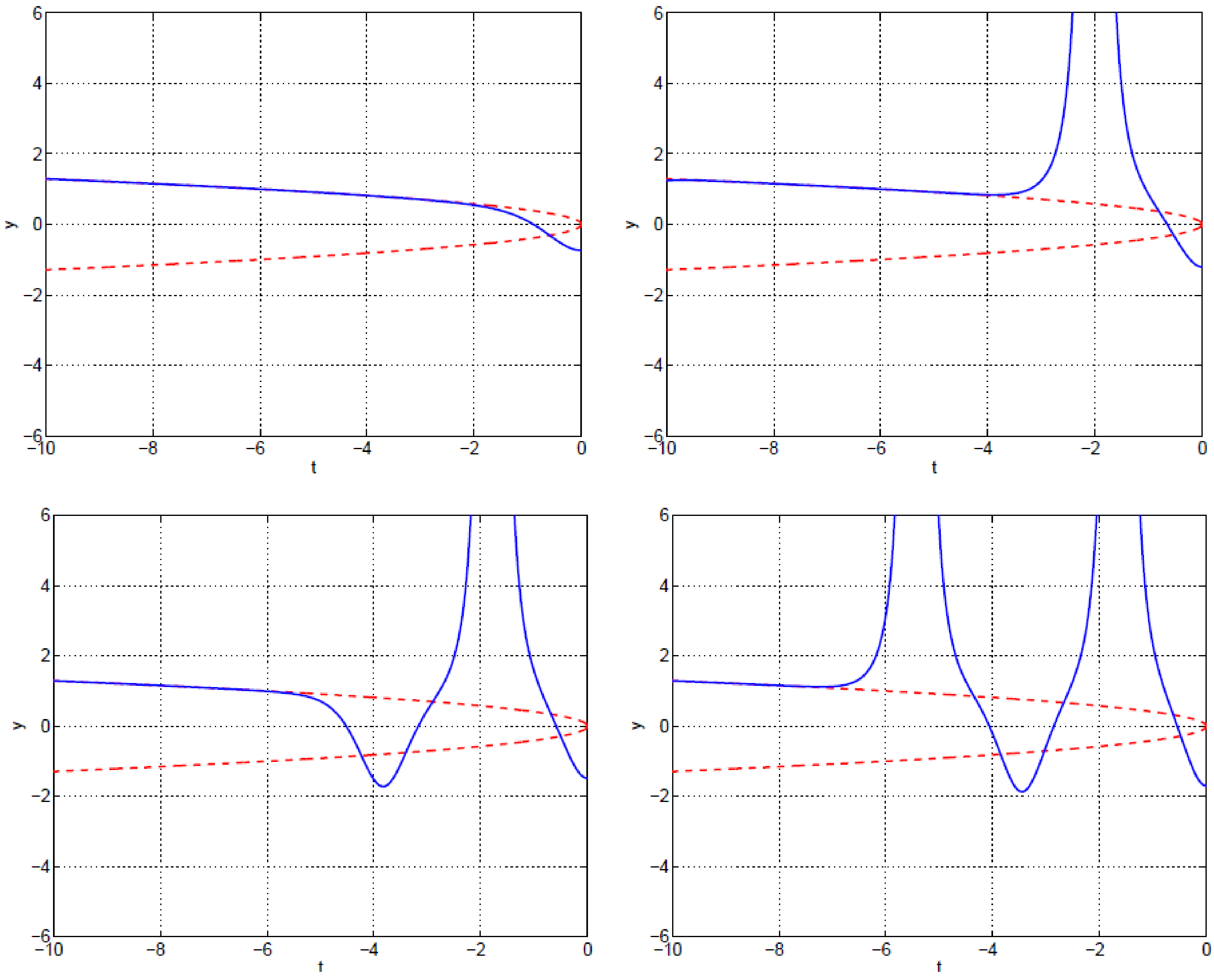
***Initial slope* is the eigenvalue, initial value *y*(0) = 0**

Tenth and eleventh separatrix (eigenfunction) solutions:



Initial slope **is the eigenvalue, initial value $y(0) = 0$**

First four separatrix solutions with 0 initial slope:



Numerical calculation of eigenvalues

(*nonlinear* semiclassical large-*n* limit)

$$y'(0) = b_n \qquad y(0) = 0$$

$$b_n \sim B_{\rm I} n^{3/5}$$

$$B_{\rm I} = 2.0921467\underline{4}$$

$$y(0) = c_n \qquad y'(0) = 0$$

$$c_n \sim C_{\rm I} n^{2/5}$$

$$C_{\rm I} = -1.030484\underline{4}$$

Analytical asymptotic calculation of eigenvalues

$$B_{\rm I} = 2\left[\frac{5\sqrt{\pi}\Gamma(5/6)}{2\sqrt{3}\Gamma(1/3)}\right]^{3/5}$$

$$C_{\rm I} = -\left[\frac{5\sqrt{\pi}\Gamma(5/6)}{2\sqrt{3}\Gamma(1/3)}\right]^{2/5}$$

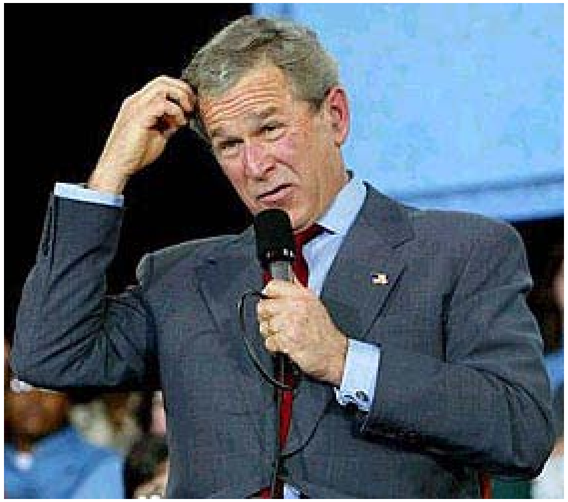
Obtained by using WKB to calculate the large eigenvalues of the

cubic PT-symmetric Hamiltonian

$$H = \frac{1}{2}p^2 + 2ix^3$$

Painlevé I corresponds to ε = 1

(Do you remember the cubic *PT*-symmetric Hamiltonian?!)



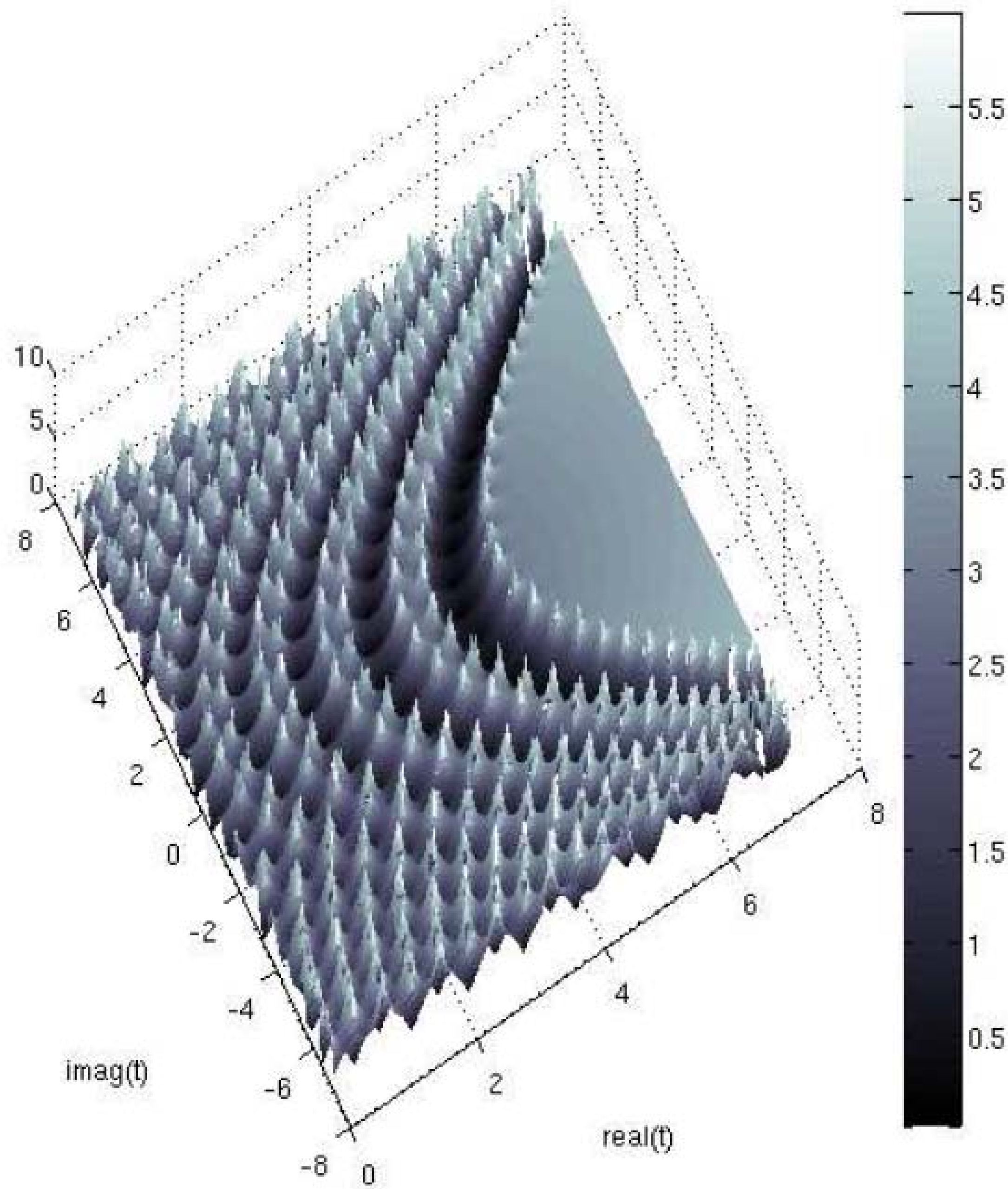
Analytical asymptotic calculation of eigenvalues

Multiply Painlevé I equation by *y'*(*t*) and integrate from $t = 0$ to $t = x$:

$$H \equiv \frac{1}{2}[y'(x)]^2 - 2[y(x)]^3 = \frac{1}{2}[y'(0)]^2 - 2[y(0)]^3 + I(x),$$

where $$I(x) = \int_0^x dt\, t y'(t).$$

If we take $|x|$ large at an angle of $\pi/4$, $I(x) \to 0$, and we get the ***PT***-symmetric Hamiltonian for $\varepsilon = 1$.



10
5
0
8
6
4
2
0
-2
-4
-6
-8
imag(t)
0
2
4
6
8
real(t)
5.5
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5

(2) Second Painlevé transcendent

$$y''(t) = 2[y(t)]^3 + ty(t), \qquad y(0) = b, \; y'(0) = c$$

Now, both solutions

$$+\sqrt{-t/2} \text{ or } -\sqrt{-t/2}$$

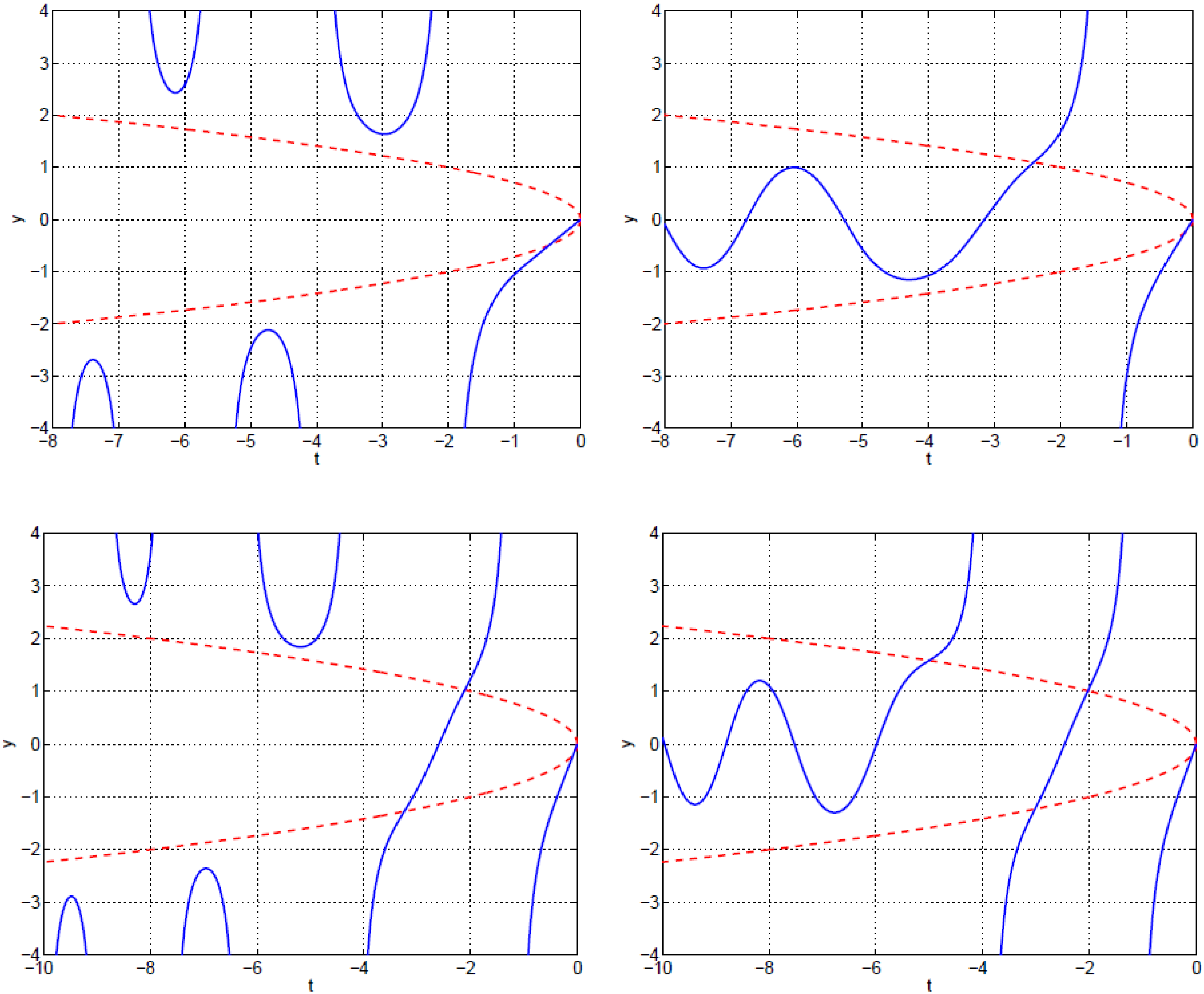
are unstable and 0 is stable.

Two types of solutions (not eigenfunctions):

Unstable

Stable

Unstable

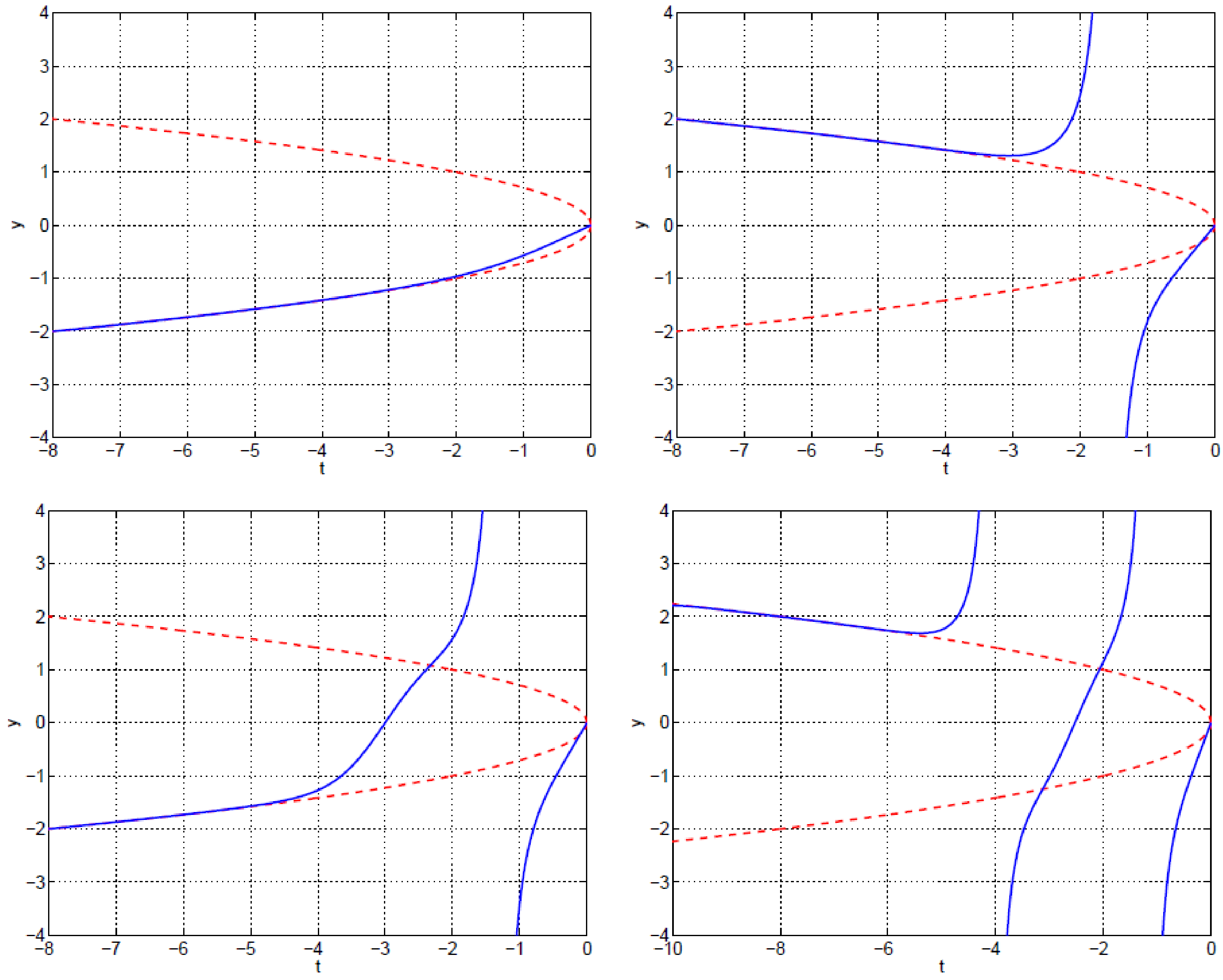


Unstable

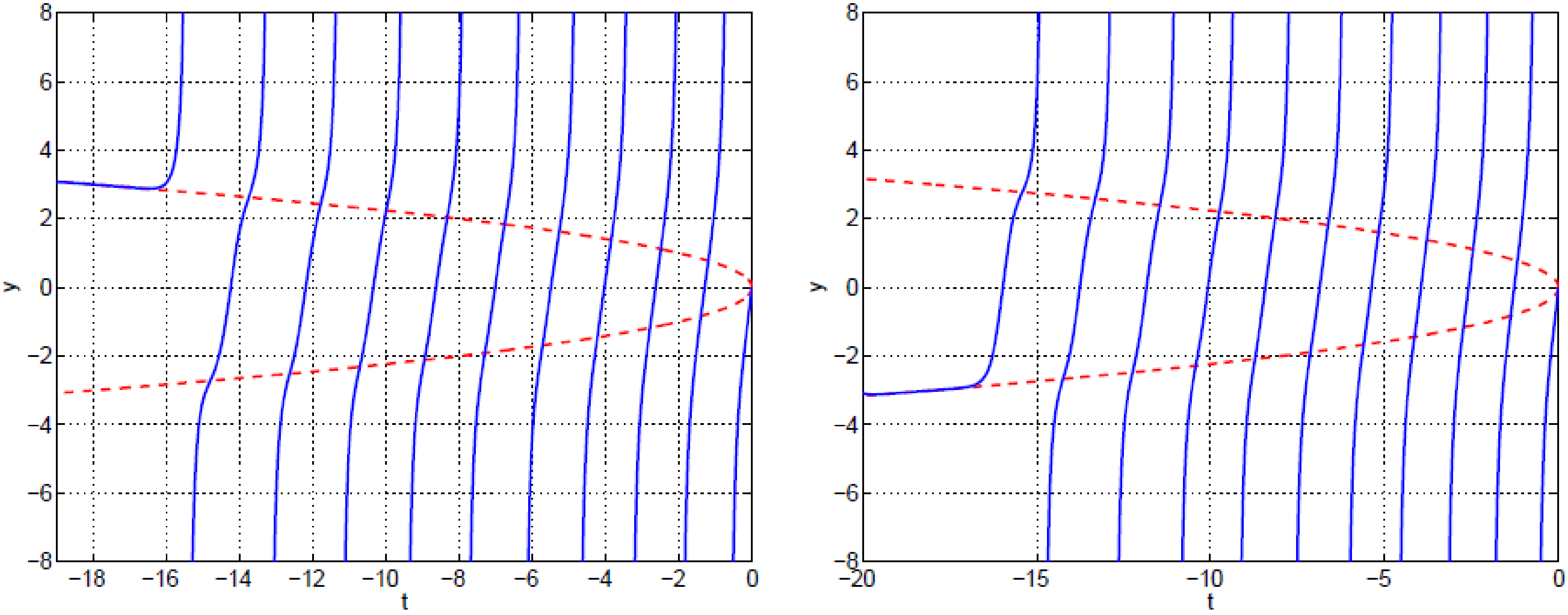
Stable

Unstable

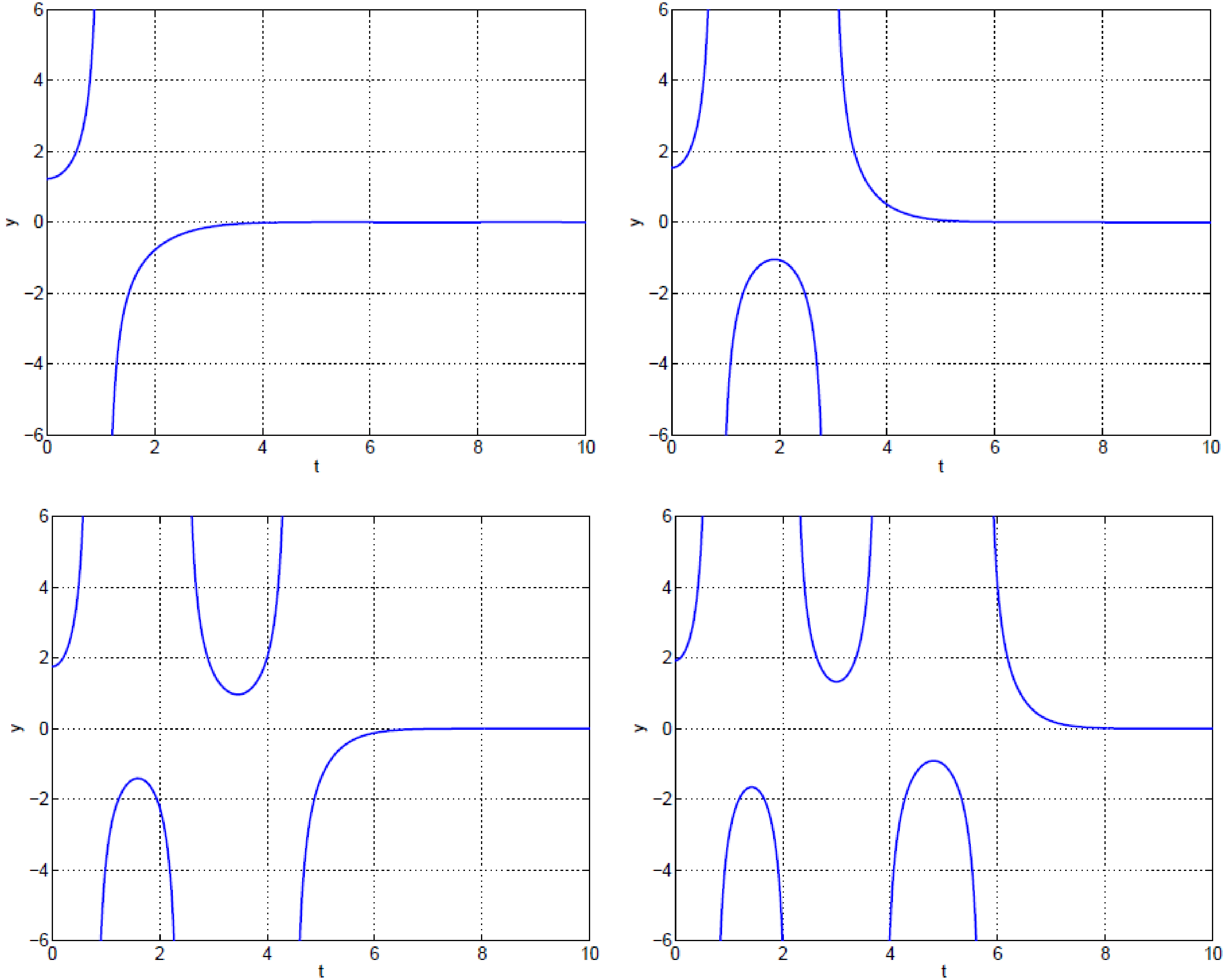
First four separatrix solutions with y(0)=0:



20^{th} and 21^{st} separatrix solutions:



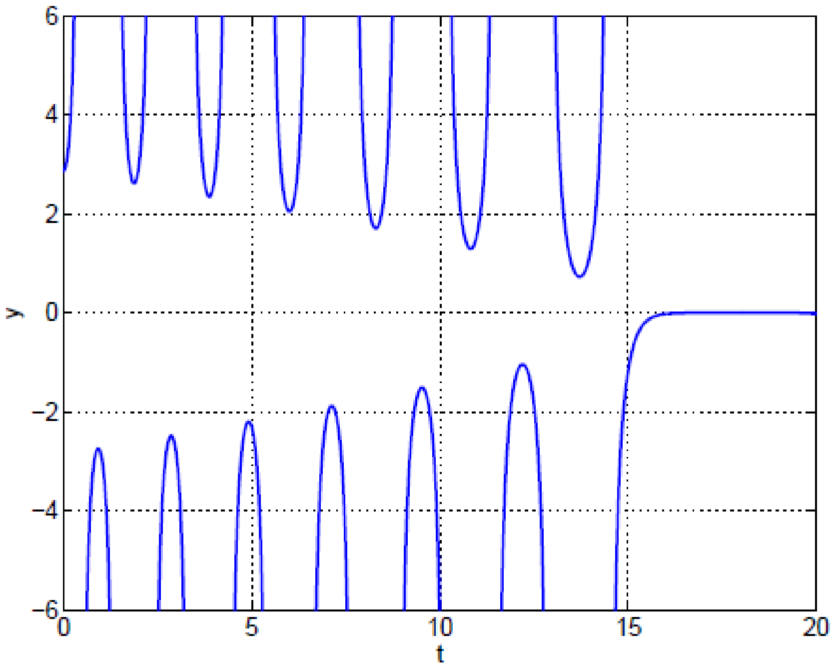
First four separatrices with vanishing initial slope $y'(0)=0$:

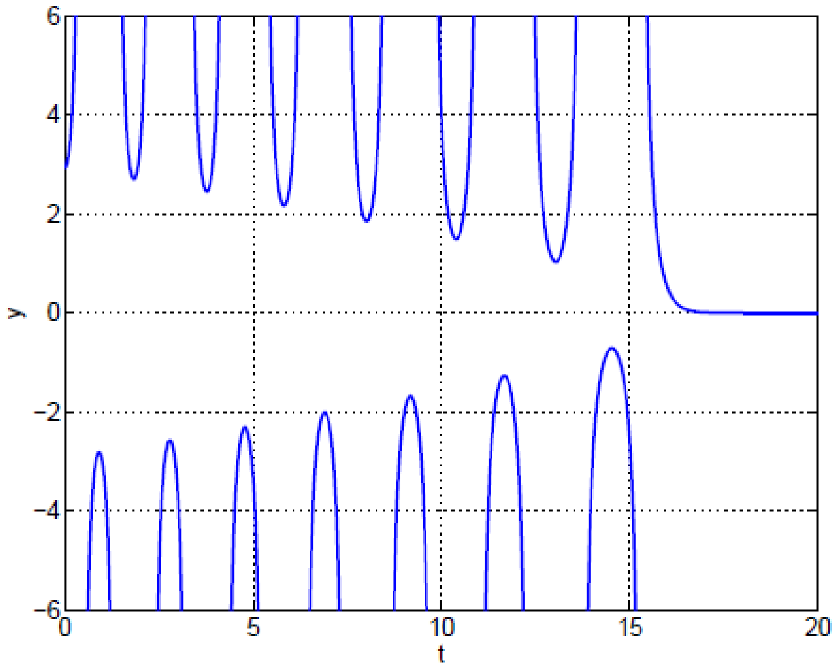


Unstable

Unstable

13th and 14th separatrices:





Numerical calculation of eigenvalues

$$y(0) = 0,\ b_n = y'(0)$$

$$c_n = y(0),\ y'(0) = 0$$

$$b_n \sim B_{\rm II} n^{2/3} \quad \text{and} \quad c_n \sim C_{\rm II} n^{1/3}$$

$$B_{\rm II} = 1.862412\underline{8} \qquad C_{\rm II} = 1.2158116\underline{5}$$

CMB and J. Komijani
J. Physics A: Math. Theor. **48**, 475202 (2015)

Analytical calculation of eigenvalues

$$B_{\mathrm{II}} = \left[3\sqrt{2\pi}\,\Gamma\left(\tfrac{3}{4}\right)/\Gamma\left(\tfrac{1}{4}\right)\right]^{2/3}$$

$$C_{II} = \left[3\sqrt{\pi}\,\Gamma\left(\tfrac{3}{4}\right)/\Gamma\left(\tfrac{1}{4}\right)\right]^{1/3}$$

Obtained by using WKB to calculate the large eigenvalues of the

quartic PT-symmetric Hamiltonian

$$H = \frac{1}{2}p^2 - \frac{1}{2}x^4$$

Painlevé II corresponds to ε = 2

(Do you remember the quartic upside-down *PT*-symmetric Hamiltonian?!)

(3) Fourth Painlevé transcendent

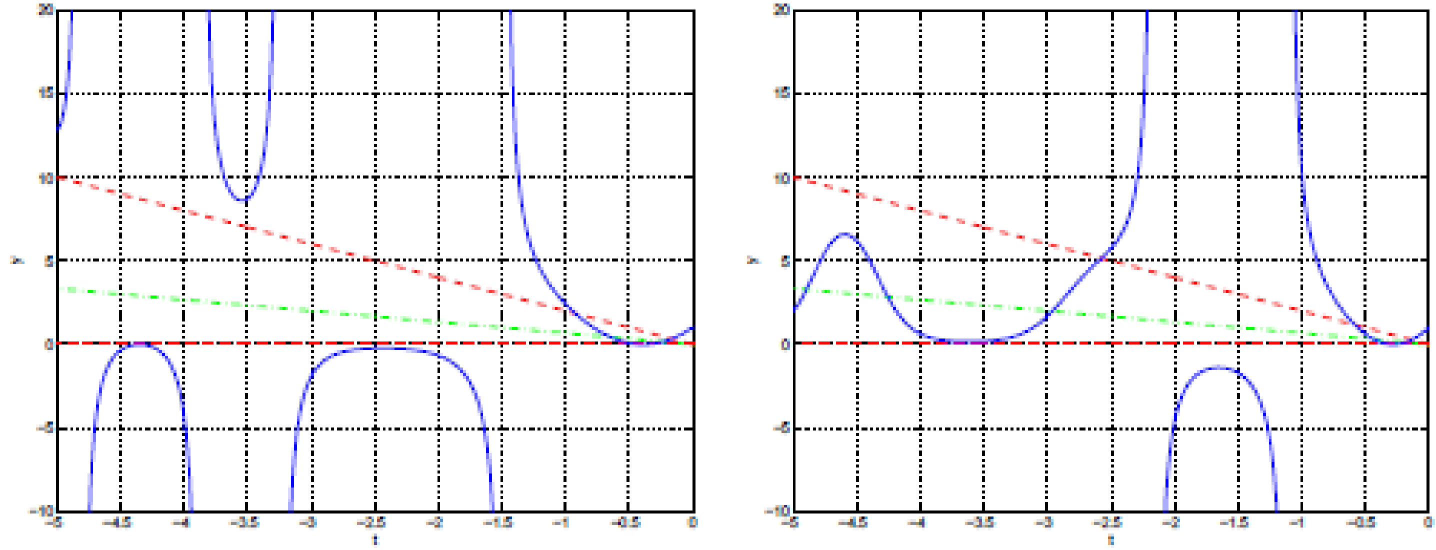
$$y(t)y''(t) = \frac{1}{2}[y'(t)]^2 + 2t^2[y(t)]^2 + 4t[y(t)]^3 + \frac{3}{2}[y(t)]^4$$

with $y(0) = c$ and $y'(0) = b$.

Two possible kinds of solutions (NOT eigenfunctions):

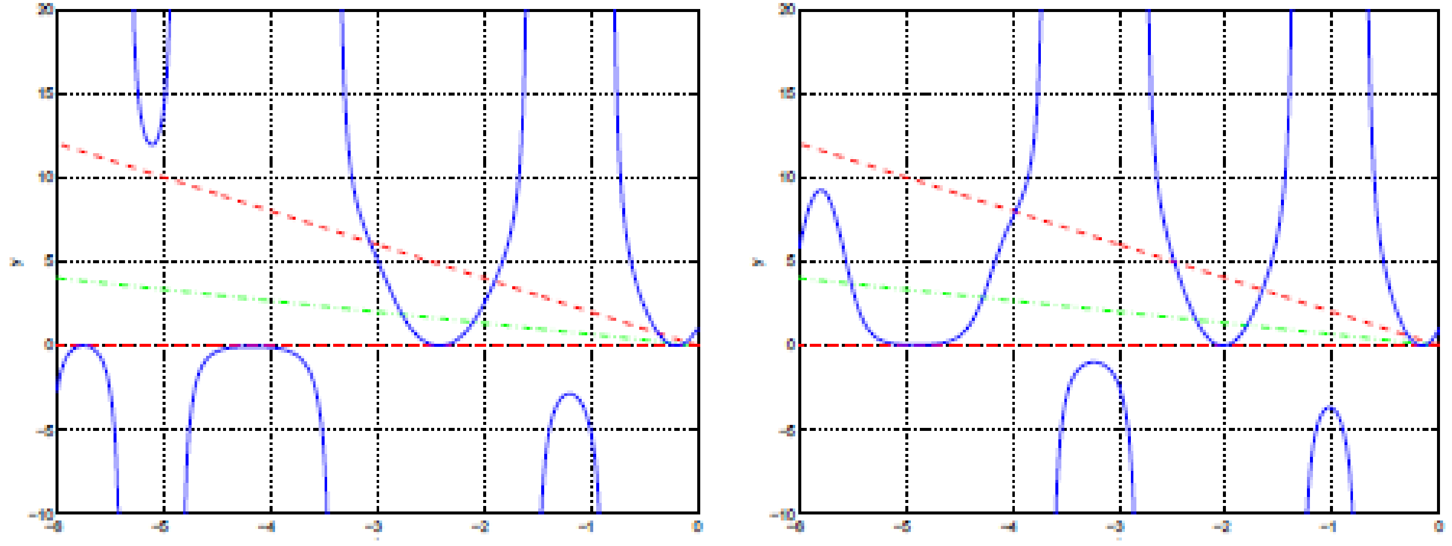
Unstable branch

Stable branch



Unstable branch

Stable branch



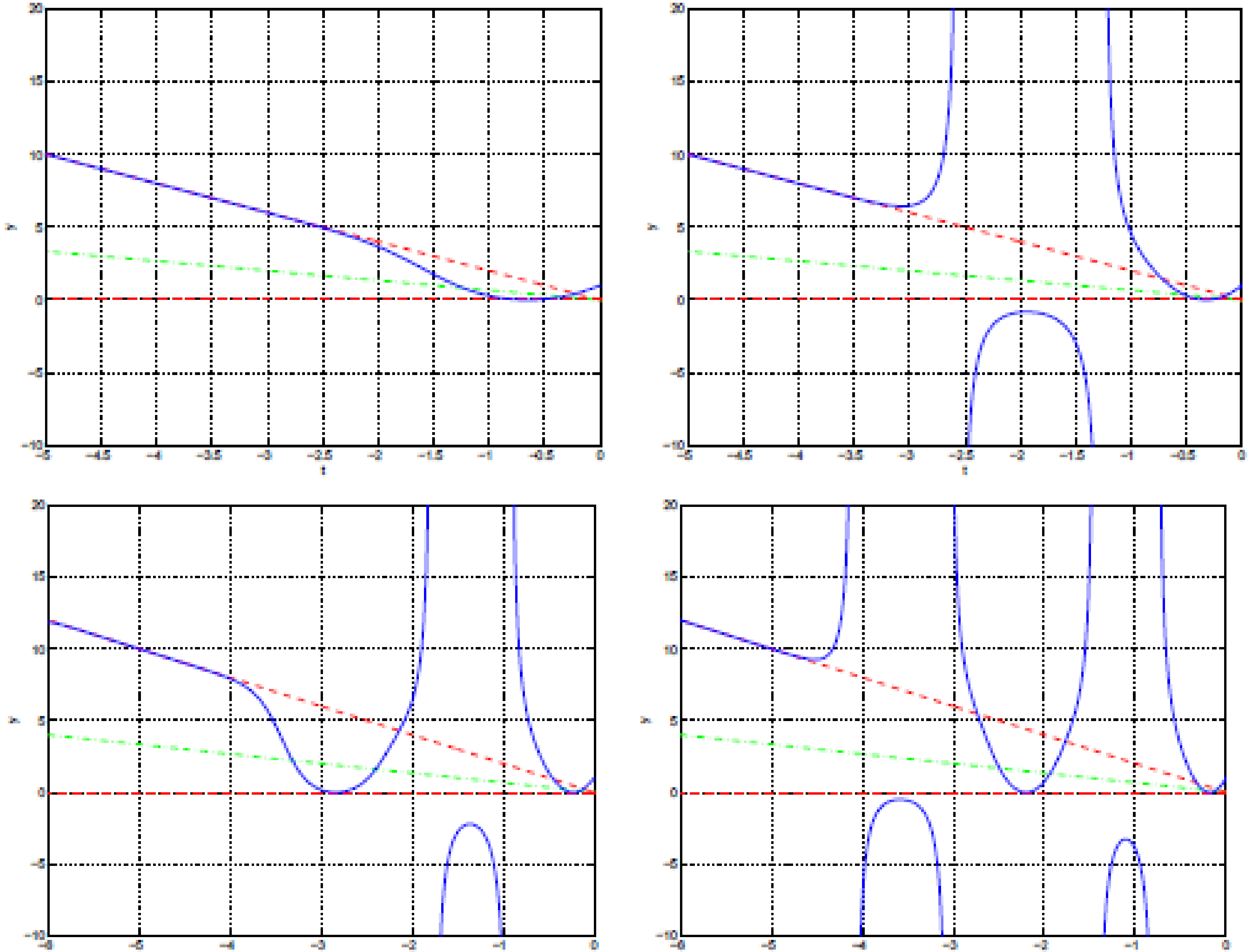
First four separatrix (eigenfunction) solutions [$y(0)=1$]:

Unstable

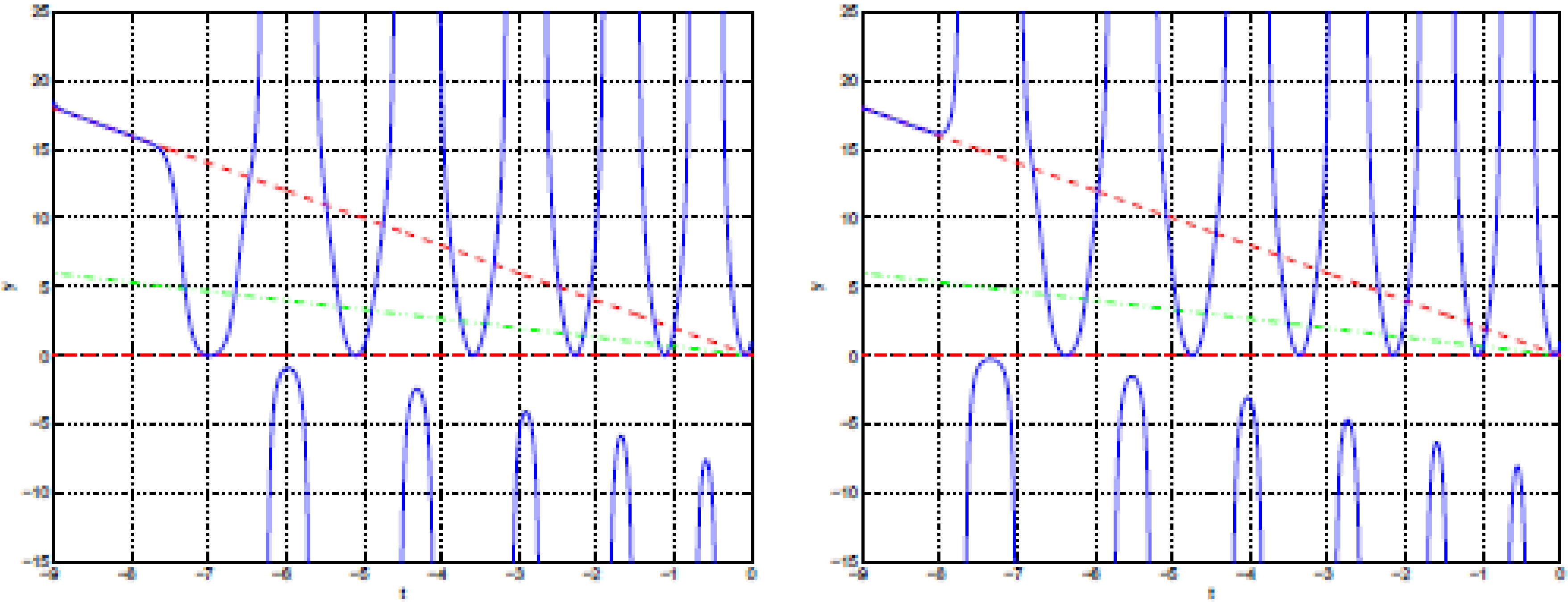
Stable

Unstable

Stable



Tenth and eleventh separatrix (eigenfunction) solutions:

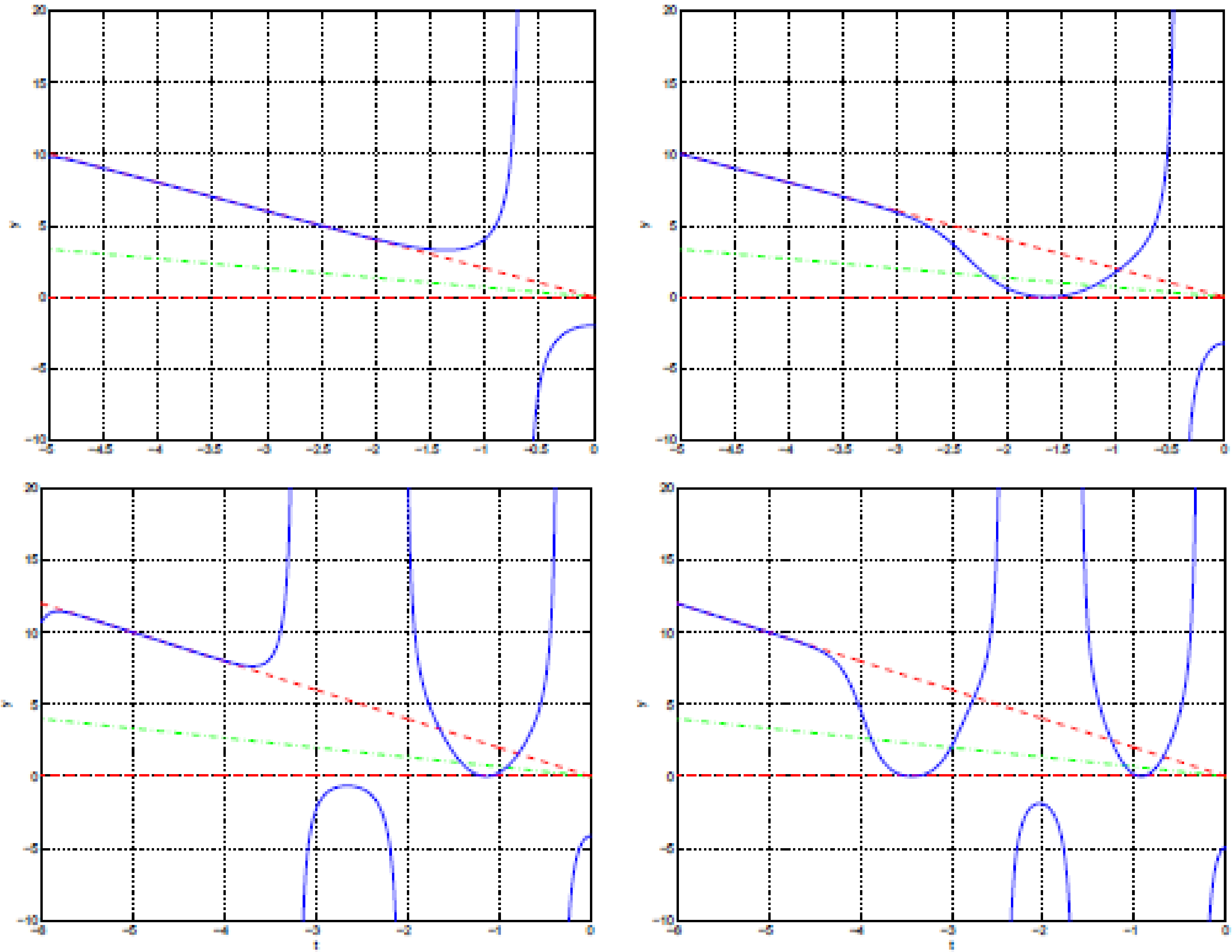


Slope **is the eigenvalue, initial value** $y(0) = 1$

First four separatrix (eigenfunction) solutions [*y*’(0)=0]:

Unstable

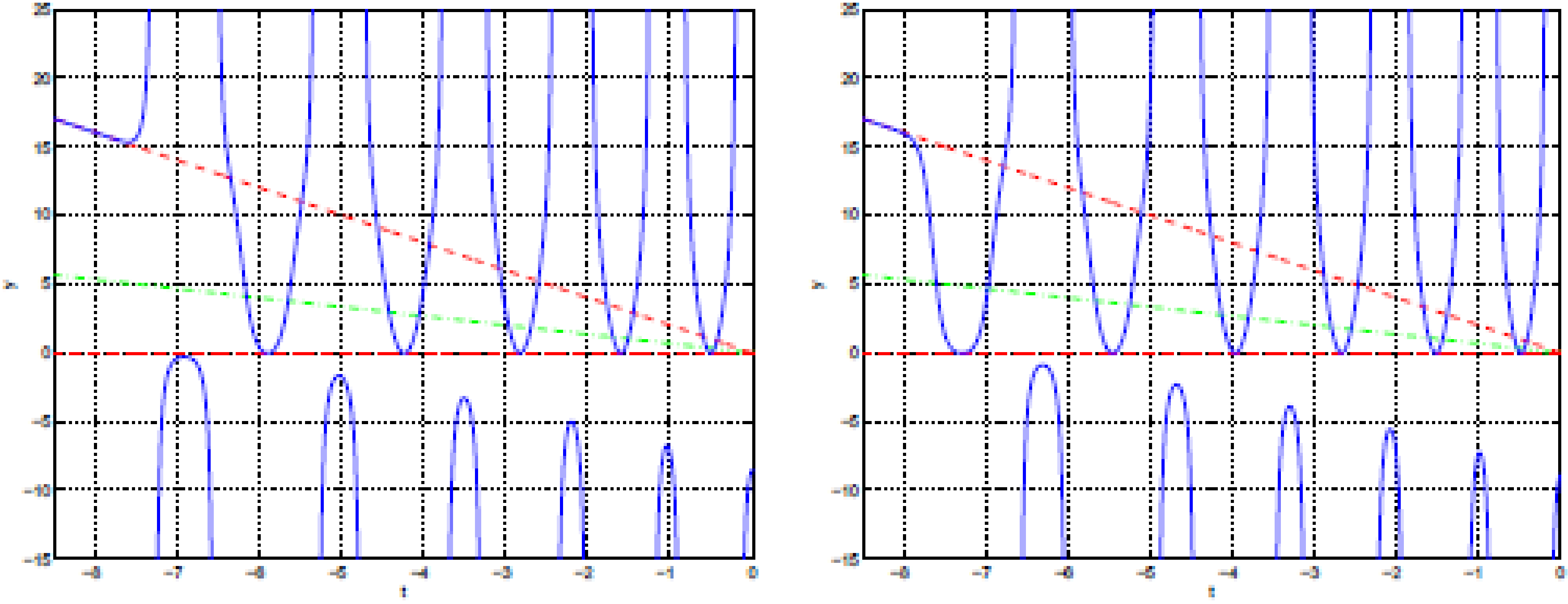
Stable



Unstable

Stable

Tenth and eleventh separatrix (eigenfunction) solutions:



$y(0)$ is the eigenvalue, initial slope is 0

Large *n* behaviour of eigenvalues: $b_n \sim B_{IV} n^{3/4}$ and $c_n \sim C_{IV} n^{1/2}$.

Numerical results using Richardson extrapolation:

$$B_{IV} = 4.25684\underline{3}.$$

$$C_{IV} = -2.62658\underline{7}$$

Analytic results using $\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{8}\hat{x}^6.$

$$B_{IV} = 2^{3/2}\left[\sqrt{\pi}\Gamma\left(\tfrac{5}{3}\right)/\Gamma\left(\tfrac{7}{6}\right)\right]^{3/4},$$

$$C_{IV} = -2\left[\sqrt{\pi}\Gamma\left(\tfrac{5}{3}\right)/\Gamma\left(\tfrac{7}{6}\right)\right]^{1/2}$$

Obtained by using WKB to calculate the large eigenvalues of the

sextic PT-symmetric Hamiltonian

The bottom line:

Painlevé I, II, and IV correspond to $\varepsilon = 1, 2,$ and 4

In general, this analysis works for huge classes of equations beyond Painlevé. For example:

$$y''(x) = \frac{2M+2}{(M-1)^2}[y(x)]^M + x[y(x)]^N$$

We hope we have opened a window to a new area of *nonlinear semiclassical* asymptotic analysis

Thanks for listening!