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On proportionally fair solutions for the divorced-parents problem

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When Dutch parents divorce, Dutch law dictates that the parental contributions to cover the financial needs of the children have to be proportionally fair. This rule is clear when parents only have common children. However, cases can be considerably more complicated, for example when parents have financial responsibilities to children from previous marriages. We show that, mathematically, this settlement problem can be modelled as a bipartite rationing problem for which a unique global proportionally fair solution exists. Moreover, we develop two efficient algorithms for obtaining this proportionally fair solution, and we show numerically that both algorithms are considerably faster than standard convex optimization techniques. The first algorithm is a novel tailor-made fixed-point iteration algorithm, whereas the second algorithm only iteratively applies simple lawsuits involving a single child and its parents. The inspiration for this latter algorithm comes from our main convergence proof in which we show that iteratively applying settlements on smaller subnetworks eventually leads to the same settlement on the network as a whole. This has significant societal importance since in practice lawsuits are often only held between two or a few parents. Moreover, our iterative algorithm is easy to understand, also by parents, legal counselors, and judges, which is crucial for its acceptance in practice. Finally, as the method provides a unique solution to any dispute, it removes the legal inequality perceived by parents. Consequently, it may considerably reduce the workload of courts because parents and lawyers can compute the proportionally fair parental contributions before bringing their case to court.

Key words: Divorced parents problem, Proportionally fair solutions, Bipartite rationing problem

1. Introduction

When two Dutch parents divorce and have common children, they both have a financial responsibility to cover their children's monthly needs, such as housing and school costs. In general, these financial needs are distributed to the parents *in proportion* to their financial capacity. This rule is clear in practice when parents have just one child, or only common children. However, cases can be considerably more complicated, in particular when the divorcing parents have financial obligations to children of previous marriages, or remarry other parents who in turn have children.

When the divorcing parents disagree on how to cover the children's needs, they can bring their case to court. By Dutch law, the distribution of the parental contributions should satisfy, roughly speaking, two properties: (i) the children's needs must be met to the maximum extent possible, and (ii), when there is financial capacity left, the parental contributions should be proportional to their individual financial capacities. Henceforth, we refer to the problem of determining parental contributions satisfying (i) and (ii) as the *divorced-parents problem (DPP)*.

The DPP can be represented as a rationing problem on a bipartite graph in which nodes represent the parents and the children, and the directed arcs from parent nodes to child nodes indicate for which children each parent is financially responsible. In particular, the financial capacities of the parents represent the scarce resources that need to be divided over several claimants, in this case the children, to meet their needs. Applying results from Moulin and Sethuraman (2013) for such rationing problems to the DPP, we conclude that a unique solution exists and can be obtained using standard convex optimization algorithms, see, e.g., Boyd and Vandenberghe (2004). However, there are three practical reasons why this is insufficient for solving the DPP in Dutch courts.

The first practical problem is operational. Typically, only two or a few parents are involved in a lawsuit, even if the actual parent-child network is considerably larger, see Jonker et al. (2020) for a number of examples. A judge may take this larger network into account when determining the parental contributions, but practical reasons compel the judge to constrain the network to a manageable size. As a consequence, the final court rulings apply only to the parents directly involved in the lawsuit, but do not extend to the full parent-child network. Therefore, a lawsuit, which reduces the unfairness in one part of the network, can increase the unfairness in another part of the network. This may lead to a second lawsuit, which may propagate a third one, leading to a sequence of lawsuits. From a mathematical point of view it is pleasing, and from a societal point of view it is important, that the parental contributions converge after such a sequence of lawsuits on subnetworks of the larger parent-child network, and that these contributions converge to the unique solution of the DPP on the complete parent-child network.

The second practical problem has a psychological background, and is related to the standard convex optimization algorithms to obtain the solution to a complete parent-child network. These algorithms are not well understood by legal counselors, parents, and judges, and thus mainly perceived as a black box. This significantly hampers the acceptance of the *unique* solution to the DPP that these algorithms provide. Particularly, when the parent-child network underlying a lawsuit is complex and large.

The third practical problem is computational. Many individual parents, mediators, and lawyers, do not have access to or are unable to use large convex optimization toolboxes, meaning that they cannot compute the solution before they bring their dispute to court. With access to a numerically fast and easy-to-understand exact solution method, however, that can easily be run on, e.g., a mobile phone, many of these lawsuits can be avoided, reducing the workload on court houses. Moreover, when it does come to a lawsuit, new information may become available during an actual trial, and with a fast and simple solution method the impact of this new information on the solution can be determined real-time.

In this paper we provide a solution method that addresses all these three practical problems at the same time. In particular,

- We prove that under mild conditions the parental contributions corresponding to any sequence of lawsuits for subnetworks converges to the unique solution of the DPP on the complete parent-child network.
- We develop two efficient exact solution methods for the DPP. The first is a novel tailor-made fixed-point iteration algorithm. The second is based on our convergence result, and is particularly easy to understand since it only requires iteratively applying simple lawsuits on subnetworks involving a single child and its parents.
- We show using numerical experiments that our methods are faster than standard convex optimization techniques. Moreover, we show that our methods find the unique proportionally fair solution to the bipartite rationing problem for any network of practical size in less than a second, even on a mobile phone. Additionally, it only requires a few lines of code to implement the methods.

We note that there exist multiple notions of proportionally fairness in the literature. While here we focus on the one in the context of the bipartite rationing problem (Moulin and Sethuraman 2013), this concept is also used in bandwidth sharing of the Internet, c.f., Kelly et al. (1998), Walton (2009), Wang et al. (2022). However, the concept of fairness that they use is not the same as ours.

The remainder of this paper is organized as follows. In Section 2 we provide further background and motivation for the practical aspects of the divorced-parents problem. In Section 3 we mathematically describe the DPP, and in Section 4 we develop a fixed-point iteration algorithm for obtaining the proportionally fair solution to the DPP. Next, in Section 5 we provide our convergence results for sequences of lawsuits on subnetworks of the parent-child network, in Section 6 we present our numerical results, and we end with a discussion in Section 7.

2. Legal and practical background of the DPP

A lawsuit about child maintenance involves, basically, three steps. First, a judge formalizes which (step-)parent is financially responsible for which child. Second, the judge determines the financial needs of each child and the financial capacity of each parent that can be used to cover the needs. The rules used in this step are partly based on case law and partly on specific circumstances such as income, cost of housing and schooling. The third step is to settle the contributions of the parents to the children, based on this information.

While many different solutions are possible, rulings of the Dutch Supreme Court and case law, cf., Jonker et al. (2020), dictate that the division should be *proportionally fair*, that is, the parental contributions should satisfy the following rules,

1. Parents only contribute to children for whom they are directly financially responsible.
2. Capacity should be used to cover the needs to the maximum extent possible. In other words, a child cannot have a shortage unless both its parents already used their full capacity.
3. Arbitrary circumstances should not have an impact on the division. That is, children from former marriages should not receive financial preference over children from a later marriage.
4. When the capacities cover the needs of a child, i.e., there is an overage, the contribution of each parent should be proportional to the parent's capacity.
5. When the capacities do not suffice to cover the needs, i.e., there is a shortage, and the siblings have different needs, the available capacity should be divided proportional to the need of each child.

We illustrate the last two rules with two simple examples for which the unique proportionally fair solution can be determined analytically.

EXAMPLE 1. Consider a parent-child network with two parents, A and B, and one common child, AB. Let d_A and d_B denote the financial capacities of the parents and b_{AB} the need of the child. Then, in the proportionally fair solution the contributions, $x_{A,AB}$ and $x_{B,AB}$, of parents A and B to child AB are given by,

$$x_{A,AB} = d_A \min\left\{\frac{b_{AB}}{d_A + d_B}, 1\right\}, \qquad \text{and} \qquad x_{B,AB} = d_B \min\left\{\frac{b_{AB}}{d_A + d_B}, 1\right\}. \tag{1}$$

That is, if the need b_{AB} of child AB exceeds the financial capacities $d_A + d_B$ of the parents, i.e., if $b_{AB} > d_A + d_B$, then $x_{A,AB} = d_A$ and $x_{B,AB} = d_B$, meaning that both parents spend their full capacity. On the other hand, if $b_{AB} \leq d_A + d_B$, then both parents spend the same proportion $\frac{b_{AB}}{d_A+d_B}$ of their financial capacity on child AB. □

EXAMPLE 2. For a parent-child network with two children, AB and BC, and one common parent, B, with financial needs, b_{AB} and b_{BC}, and capacity, d_B, respectively, the proportionally fair contributions, $x_{B,AB}$ and $x_{B,BC}$, of parent B to children AB and BC is given by

$$x_{B,AB} = b_{AB} \min\left\{\frac{d_B}{b_{AB} + b_{BC}}, 1\right\}, \qquad \text{and} \qquad x_{B,BC} = b_B \min\left\{\frac{d_B}{b_{AB} + b_{BC}}, 1\right\}. \tag{2}$$

Note that the children receive exactly their needs when $d_B \geq b_{AB} + b_{BC}$, i.e., when the financial capacity of parent B exceeds the combined needs of both children AB and BC. Otherwise, both children receive the same proportion $\frac{d_B}{b_{AB}+b_{BC}}$ of their needs. □

The cases that are brought to court are typically much more complicated than in Examples 1 and 2. The lawsuit described in Example 3 below based on an actual court case—in the bibliography indicated as Case:767 (2016)—illustrates the complexities that occur in practice.

EXAMPLE 3. Parents A and B have a case about the contribution to their common (three) children AB, see Fig. 1 for an overview of the topology of the parent-child network. At the date of the case, A is living together with C with common child AC. Parent C has also another child CD from a former marriage with D. Parent B, the other parent involved in the case, is remarried to E, who already had a child EF with F; F is not included in the description of the lawsuit for reasons not mentioned in the lawsuit. Parent B accepts responsibility for child EF while E accepts responsibility for AB.

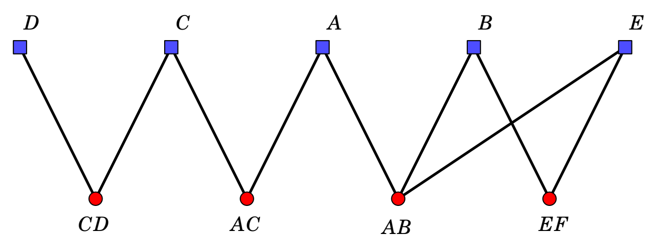

Figure 1 The parent-child network of the lawsuit described in Example 3. Parents are represented as squares at the top; children as circles at the bottom.

Table 1 The outcome of the actual court case described in Example 3. The bold numbers in the matrix indicate the parental contributions in euros per month as determined by court order; the proportionally fair contributions are given in parentheses. On top of the matrix are the financial needs of the children, to the left the financial capacities of the parents, and to the right the overages of the parents, i.e., the financial capacity they have left after paying the contributions to their children.

		Financial needs children				
		AB	AC	CD	EF	
Capacities parents		1563	728	790	520	Overages
A	3795	**842** (*817*)	**336** (*545*)	-	-	2617 (2433)
B	2545	**550** (*545*)	-	-	**285** (*380*)	1710 (1620)
C	1374	-	**392** (*183*)	**527** (*372*)	-	455 (819)
D	1355	-	-	**263** (*418*)	-	1092 (937)
E	939	**171** (*201*)	-	-	**235** (*140*)	533 (598)

Table 1 shows, besides the financial capacities of the parents and the financial needs of the children, the parental contributions as ruled by the court and the proportionally fair parental contributions (in parentheses). A direct comparison of both outcomes shows striking differences. In particular, parent C needs to contribute 364 euros per month more than what is proportionally fair, whereas parents A and D need to contribute, respectively, 184 and 175 euros per month less than in the proportionally fair solution. □

From Example 3 and other examples investigated by Jonker et al. (2020) it is apparent that judges attempt to find proportionally fair solutions consistent with the rulings of the Dutch Supreme Court, but in practice accept (very rough) approximations due to the complexity of the computations. This has numerous negative consequences. First, for more or less similar situations, the parental contributions as ruled by different courts can differ significantly, thereby leading to legal inequality for parents. Second, when ex-spouses perceive the settlements as 'unfair' or 'arbitrary', they can (and sometimes do) start new costly and lengthy court cases. Formulated in plain human terms: there is much grief, frustration, and anger involved in this process. Third, the computation by hand or Excel of even an approximately fair distribution is time-consuming, and hence places a considerable burden on the capacity of the courts. As a further consequence of the lengthy procedures, contributions of the parents are not updated even when the network changes considerably, for instance due to new born children. In summary, there is a clear need for a fast and widely accepted exact solution method to compute the proportionally fair division satisfying the rulings of the Dutch Supreme Court.

In the remainder of the paper we concentrate on the mathematical aspects of the DPP, and methods to find the proportionally fair solution. Even though we present our results in terms of the DPP, we note that these apply to the general proportionally fair bipartite rationing problem.

3. The divorced-parents problem

Let $\mathcal{N} = (V, \mathcal{E}, d, b)$ denote a bipartite parent-child network with $V = \mathcal{P} \cup \mathcal{C}$ such that each node $i \in \mathcal{P}$ corresponds to a parent and every node $j \in \mathcal{C}$ to a child. The network $\mathcal{N}$ has a directed arc $(i,j) \in \mathcal{E} \subset \mathcal{P} \times \mathcal{C}$ from node $i \in \mathcal{P}$ to node $j \in \mathcal{C}$ if and only if parent i is financially responsible for child j. Parents have financial capacities $d_i > 0$, $i \in \mathcal{P}$, to cover the financial needs $b_j > 0$, $j \in \mathcal{C}$, of the children. We assume that each child $j \in \mathcal{C}$ has at least two parents, but can have more in case of step-parents as in Example 3. Finally, we assume that the network $\mathcal{N}$ consists of one component, for otherwise we study each component in separation. We will call such a network $\mathcal{N}$ *connected*.

For every arc $(i,j) \in \mathcal{E}$, let $x_{ij} \in \mathbb{R}_+$ denote the contribution that parent i pays to child j. Define the set of feasible parental contributions x as

$$X := \left\{ x \in \mathbb{R}_+^{|\mathcal{E}|} : \sum_{j \in \mathcal{C}(i)} x_{ij} \leq d_i, \ \forall\, i \in \mathcal{P}; \ \sum_{i \in \mathcal{P}(j)} x_{ij} \leq b_j \ \forall j \in \mathcal{C} \right\},$$

where $\mathbb{R}_+ = [0, \infty)]$, $\mathcal{C}(i) := \{j \in \mathcal{C} : (i,j) \in \mathcal{E}\}$ represents the children of parent $i \in \mathcal{P}$ and $\mathcal{P}(j) := \{i \in \mathcal{P} : (i,j) \in \mathcal{E}\}$ the parents of child $j \in \mathcal{C}$. Thus, $x \in X$ if and only if the total parental contribution of each parent $i \in \mathcal{P}$ does not exceed the financial capacity d_i, and each child $j \in \mathcal{C}$ does not receive more than the financial needs b_j. Observe, for the moment, that we do not require that the needs of the children are exactly met since this may be infeasible, for example if $\sum_{i\in\mathscr{P}} d_i < \sum_{j\in\mathscr{C}} b_j$.

For every parental contribution $x \in X$ we define the *overage* $O_i(x)$ of parent $i \in \mathscr{P}$ as

$$O_i(x) := d_i - \sum_{j\in\mathscr{C}(i)} x_{ij}.$$

That is, the overage $O_i(x)$ of parent i is how much financial capacity parent i has left after paying x_{ij} to children $j \in \mathscr{C}(i)$. Observe that $x \in X$ implies that $O_i(x) \geq 0$. Analogously, we define the *shortage* $S_j(x)$ of child $j \in \mathscr{C}$ as

$$S_j(x) := b_j - \sum_{i\in\mathscr{P}(j)} x_{ij},$$

where $S_j(x) \geq 0$ for all $j \in \mathscr{C}$ if $x \in X$.

Clearly, the rulings of the Dutch Supreme Court described in Section 2 imply that a proportionally fair solution $\bar{x} \in X$ should minimize the total shortages of the children, that is,

$$\bar{x} \in \operatorname{arg\,min}\left\{\sum_{j\in\mathscr{C}} S_j(x) : x \in X\right\}. \tag{3}$$

This problem can be interpreted as a max-flow problem where the parental contributions x represent the flow in the network, the parent nodes $\mathscr{P}$ represent the supply nodes with supply d_i, $i \in \mathscr{P}$, and the child nodes $\mathscr{C}$ represent the demand nodes with demand b_j, $j \in \mathscr{C}$.

If the max-flow problem only has a single optimal solution $\bar{x}$ for which the shortages of all children and overage of all parents are zero, i.e., $S_j(\bar{x}) = 0$ for all $j \in \mathscr{C}$ and $O_i(\bar{x}) = 0$ for all $i \in \mathscr{P}$, then we call the network *tight*. In general, however, a network is not tight and can be divided into three subnetworks, see also, Ahuja et al. (1993), Bochet et al. (2012) and Moulin and Sethuraman (2013).

1) An overage subnetwork that allows no shortages and the number of parents with an overage is maximal.

2) A shortage subnetwork, which is essentially the reverse of an overage network: there are no overages and the number of children with a shortage is maximal.

3) A tight subnetwork, consisting of the parents and children not in the overage or shortage subnetworks.

In the remainder of this paper, we focus on overage networks only. Shortage networks can be solved by the same principles by reversing the flows, and tight networks can be solved by simply solving the max-flow problem in Eq. (3).

DEFINITION 1. A parent-child network $\mathscr{N}$ is called an *overage* network if for every subset $\mathscr{C}' \subset \mathscr{C}$ of children, it holds that

$$\sum_{i\in\mathscr{P}(\mathscr{C}')} d_i > \sum_{j\in\mathscr{C}'} b_j.$$

Here, $\mathscr{P}(\mathscr{C}') := \{i \in \mathscr{P} : (i,j) \in \mathscr{E} \text{ for some } j \in \mathscr{C}'\}$, i.e., $\mathscr{P}(\mathscr{C}')$ is the set of parents with financial responsibility for at least one child in $\mathscr{C}'$.

Within an overage network, we can sensibly introduce the notion of 'budget' $B_{ij}(x)$ which is the capacity some parent $i \in \mathscr{P}$ has available for child $j \in \mathscr{C}(i)$ after meeting the obligations to all other children $\mathscr{C}(i)\backslash\{j\}$.

DEFINITION 2. Given a set of contributions $x \in X$, the *budget* $B_{ij}(x)$ of parent $i \in \mathscr{P}$ for child $j \in \mathscr{C}(i)$ is defined as

$$B_{ij}(x) := d_i - \sum_{k \in \mathscr{C}(i)\backslash\{j\}} x_{ik} = O_i(x) + x_{ij}.$$

Supposing that the contributions x are such that $B_{ij}(x) > 0$ for all $i \in \mathscr{P}$, $j \in \mathscr{C}(i)$, the overages should be divided proportionally fair.

DEFINITION 3. A set of contributions x is *feasible* when $x \in \mathscr{X} := \{x \in X : \sum_{i \in \mathscr{P}(j)} x_{ij} = b_j, \forall j \in \mathscr{C}\}$. A set $\bar{x}$ is *proportionally fair* when $\bar{x}$ is feasible and such that for every child $j \in \mathscr{C}$, there exists a constant $\delta_j > 0$ such that $\bar{x}_{ij}/B_{ij}(\bar{x}) = \delta_j$ for all $i \in \mathscr{P}_j$.

Thus, in a proportionally fair solution $\bar{x}$ all needs of the children are satisfied, and if parents i and k have a common child j, then the ratio between the contributions and budgets for both parents are equal, i.e., $\bar{x}_{ij}/B_{ij}(\bar{x}) = \bar{x}_{kj}/B_{kj}(\bar{x}) = \delta_j$. For instance, for the proportionally fair solution in Table 1 of Example 3 we have that $\delta_{AB} = 0.25, \delta_{AC} = 0.18, \delta_{CD} = 0.31$, and $\delta_{EF} = 0.19$.

It turns out that this proportionality condition has two equivalent characterizations.

PROPOSITION 1. *Let $\mathscr{N} = (\mathscr{V}, \mathscr{E}, d, b)$ denote a parent-child* overage *network. Let $x \in \mathscr{X}$. Then, the following three proportionality conditions on x are equivalent.*

(i) For every child $j \in \mathscr{C}$, there exists a constant $\delta_j > 0$ such that

$$\frac{x_{ij}}{B_{ij}(x)} = \delta_j, \qquad \text{for all } i \in \mathscr{P}_j.$$

(ii) For every child $j \in \mathscr{C}$, there exists a constant $\gamma_j > 0$ such that

$$\frac{O_i(x)}{B_{ij}(x)} = \gamma_j, \qquad \text{for all } i \in \mathscr{P}_j.$$

(iii) For every child $j \in \mathscr{C}$, there exists a constant $\beta_j > 0$ such that

$$\frac{x_{ij}}{O_i(x)} = \beta_j, \qquad \text{for all } i \in \mathscr{P}_j.$$

Proof. We show that (i)–(iii) are equivalent by rewriting $\frac{O_i(x)}{B_{ij}(x)}$, $j \in \mathscr{C}$, $i \in \mathscr{P}(j)$. Using the definitions of overage and budget, it follows that $B_{ij}(x) = O_i(x) + x_{ij}$, and thus

$$\frac{O_i(x)}{B_{ij}(x)} = \frac{O_i(x)}{O_i(x) + x_{ij}} = \frac{O_i(x) + x_{ij} - x_{ij}}{O_i(x) + x_{ij}} = 1 - \frac{x_{ij}}{O_i(x) + x_{ij}} = 1 - \frac{x_{ij}}{B_{ij}(x)}.$$

Hence, if $\frac{O_i(x)}{B_{ij}(x)}$ is constant for all $i \in \mathscr{P}(j)$, then so is $\frac{x_{ij}}{B_{ij}(x)}$ for all $i \in \mathscr{P}(j)$, and thus (i) and (ii) are equivalent. Note that $O_i(x) > 0$, for otherwise, parent i has no overage, and hence would be in the tight subnetwork. The equivalence with (iii) follows from the fact that

$$\frac{B_{ij}(x)}{x_{ij}} = \frac{O_i(x) + x_{ij}}{x_{ij}} = 1 + \frac{O_i(x)}{x_{ij}},$$

and thus $\frac{x_{ij}}{O_i(x)}$ is constant for all $i \in \mathscr{P}(j)$, if $\frac{x_{ij}}{B_{ij}(x)}$ is. Finally, note that by assumption all denominators of the quotients in this proposition are strictly positive and thus the quotients are well-defined. □

Proposition 1 shows that in a proportionally fair solution there is not only proportionality between parental contributions and budgets, but also between overages and budgets, and between contributions and overages. Moulin and Sethuraman (2013) show that such a proportionally fair solution exists and is unique. Therefore, we refer to $\bar{x}(\mathscr{N})$ as *the* proportionally fair solution for network $\mathscr{N}$, and we omit the dependence on $\mathscr{N}$ if it is clear from the context.

4. Computation of the proportionally fair solution

In this section we provide our first algorithm to compute the proportionally fair solution $\bar{x}$ of a parent-child overage network $\mathscr{N}$. In such a proportionally fair solution $\bar{x}$, by Proposition 1, for each child $j \in \mathscr{C}$ the parental contributions $\bar{x}_{ij}$ are proportional to the overages $O_i(\bar{x})$ of the parents $i \in \mathscr{P}(j)$. As a result, the parental contributions $\bar{x}$ are uniquely determined by the overages $O(\bar{x})$.

LEMMA 1. *In an overage parent-child network $\mathscr{N}$, any proportionally fair solution $\bar{x}$ satisfies*

$$\bar{x}_{ij} = \frac{O_i(\bar{x})}{\sum_{k \in \mathscr{P}(j)} O_k(\bar{x})} b_j, \qquad \forall j \in \mathscr{C}, i \in \mathscr{P}(j). \tag{4}$$

Proof. Let $\bar{x}$ be a proportionally fair solution and let $j \in \mathscr{C}$ be given. By Proposition 1 there exists a constant $\beta_j > 0$ such that $\bar{x}_{ij} = \beta_j O_i(\bar{x})$ for all $i \in \mathscr{P}(j)$. Moreover, since $\bar{x} \in \mathscr{X}$, it holds that $\sum_{i \in \mathscr{P}(j)} \bar{x}_{ij} = b_j$ for all $j \in \mathscr{C}$, and thus $\beta_j \sum_{i \in \mathscr{P}(j)} O_i(\bar{x}) = b_j$, from which it follows that

$$\beta_j = \frac{b_j}{\sum_{i \in \mathscr{P}(j)} O_i(\bar{x})}, \qquad j \in \mathscr{C}.$$

The desired result follows from substituting this expression in $\bar{x}_{ij} = \beta_j O_i(\bar{x})$. □

Lemma 1 shows that to determine a proportionally fair solution $\bar{x}$ in an overage network it suffices to determine the overages $O(\bar{x})$ of the parents. Implicitly, the parental contributions $\bar{x}$ are completely determined by $O(\bar{x})$. The question that remains is whether we can derive a condition on the overages $O(\bar{x})$ such that $O_i(\bar{x}) + \sum_{j \in \mathscr{C}(i)} \bar{x}_{ij} = d_i$ holds for all $i \in \mathscr{P}$ if $\bar{x}_{ij}$ is determined by (4) in Lemma 1. Lemma 2 below provides such a condition.

LEMMA 2. *In an overage network, any proportionally fair solution $\bar{x}$ satisfies*

$$O_i(\bar{x}) = d_i\Big(1 + \sum_{j\in\mathscr{C}(i)} \frac{b_j}{\sum_{k\in\mathscr{P}(j)} O_k(\bar{x})}\Big)^{-1}, \qquad \forall\, i \in \mathscr{P}. \tag{5}$$

Proof. By definition, $O_i(\bar{x}) = d_i - \sum_{j\in\mathscr{C}(i)} \bar{x}_{ij}$ for all $i \in \mathscr{P}$. Substituting the expression for $\bar{x}_{ij}$ of Lemma 1 yields

$$d_i = O_i(x) + \sum_{j\in\mathscr{C}(i)} b_j \frac{O_i(\bar{x})}{\sum_{k\in\mathscr{P}(j)} O_k(\bar{x})} = O_i(\bar{x})\Big(1 + \sum_{j\in\mathscr{C}(i)} \frac{b_j}{\sum_{k\in\mathscr{P}(k)} O_k(\bar{x})}\Big),$$

and the desired result follows from rewriting this expression. □

In Lemma 2, the factor $\Big(1+\sum_{j\in\mathscr{C}(i)} \frac{b_j}{\sum_{k\in\mathscr{P}(j)} O_k(\bar{x})}\Big)^{-1} \in (0,1)$ represents the fraction of parent i's financial capacity d_i that will be left after the parental contributions $\bar{x}_{ij}$ to children $j \in \mathscr{C}(i)$ are paid. Intuitively, if

$$O_i(\bar{x}) < d_i\Big(1 + \sum_{j\in\mathscr{C}(i)} \frac{b_j}{\sum_{k\in\mathscr{P}(j)} O_k(\bar{x})}\Big)^{-1},$$

then the current overage $O_i(\bar{x})$ is too low for a proportionally fair solution, which has to be resolved by decreasing the parental contributions of parent i and increasing those of the other parents.

Based on the condition in Lemma 2, we consider the following transformation function T. Let $\mathbb{R}_{++} = (0,\infty)$.

DEFINITION 4. We define the transformation function $T : \mathbb{R}_{++}^{|\mathscr{P}|} \to \mathbb{R}_{++}^{|\mathscr{P}|}$ as $T(y) = (T_1(y), \ldots, T_{|\mathscr{P}|}(y))$, where for every $i \in \mathscr{P}$,

$$T_i(y) = d_i\Big(1 + \sum_{j\in\mathscr{C}(i)} \frac{b_j}{\sum_{k\in\mathscr{P}(j)} y_k}\Big)^{-1}, \qquad y \in \mathbb{R}_{++}^{|\mathscr{P}|}.$$

The reason for defining this transformation function T is that the overages $O(\bar{x})$ in a proportionally fair solution $\bar{x}$ are a fixed point of T. That is, for $\bar{y} = O(\bar{x})$, it holds that $\bar{y} = T(\bar{y})$. We will show in Theorem 1 that $O(\bar{x})$ can be obtained by iteratively applying T. First, however, we discuss properties of the transformation function T. For this, we write $y' < y$ if and only if $y'_i < y_i$ for all components $i \in \mathscr{P}$, and likewise, $y' \le y$ if and only if $y'_i \le y_i$ for all $i \in \mathscr{P}$.

PROPOSITION 2. *Consider the transformation function T from Definition 4. Then,*

(i) for every $y \in \mathbb{R}_{++}^{|\mathscr{P}|}$, we have $T(y) \in \mathbb{R}_{++}^{|\mathscr{P}|}$.

(ii) the function T is continuous.

(iii) for every $y, y' \in \mathbb{R}_{++}^{|\mathscr{P}|}$, if $y \le y'$, then $T(y) \le T(y')$, and if $y < y'$, then $T(y) < T(y')$.

Proof. Trivial. ☐

Note that Proposition 2 (i) and (iii) imply that if $y \in \mathbb{R}_{++}^{|\mathscr{P}|}$ and $T(y) < y$, then $T(T(y)) < T(y) < y$. That is, under the starting condition $T(y^0) < y^0$, iteratively applying T yields a decreasing sequence that is converging because the sequence is bounded from below by zero. This observation motivates our fixed-point iteration algorithm: we define $y^0 := d$ and $y^n := T(y^{n-1})$ for $n \in \mathbb{N}$. We note that $y^0 = d$ corresponds to a situation in which the overages of all parents equal their financial capacities, and thus the children receive no contributions. Obviously, this does not correspond to a proportionally fair solution. However, by iteratively applying the transformation function T, the overages of the parents will iteratively decrease, and for the fixed point $\bar{y} = \lim_{n\to\infty} y^n$ the financial needs of the children will be exactly satisfied.

THEOREM 1. *In a parent-child overage network, let the sequence* $\{y^n\}_{n=0}^{\infty} \subset \mathbb{R}_{++}^{|P|}$ *be defined as*

$$y^n := T(y^{n-1}), \qquad n \in \mathbb{N},$$

with $y^0 := d$ *and the transformation function* T *as defined in Definition 4. Then,* y^n *converges to a fixed point* $\bar{y}$ *of* T*, and the proportionally fair solution* $\bar{x}$ *to the divorced-parents problem can be computed by*

$$\bar{x}_{ij} = \frac{\bar{y}_i}{\sum_{k\in\mathscr{P}(j)} \bar{y}_k} b_j, \qquad \forall j \in \mathscr{C}, i \in \mathscr{P}(j). \tag{6}$$

Proof. Let $y^0 := d$, and define $y^n := T(y^{n-1})$ for $n \in \mathbb{N}$. We will prove by induction that $y^n < y^{n-1}$ for all $n \in \mathbb{N}$. For $n = 1$, this is true since for every $i \in \mathscr{P}$,

$$y_i^1 = T_i(y^0) = d_i\Big(1 + \sum_{j\in\mathscr{C}(i)} \frac{b_j}{\sum_{k\in\mathscr{P}(j)} y_k^0}\Big)^{-1} < d_i = y_i^0.$$

For arbitrary $n \in \mathbb{N}$, we have $y^{n+1} = T(y^n) < T(y^{n-1}) = y^n$, since by the induction hypothesis $y^n < y^{n-1}$ and thus $T(y^n) < T(y^{n-1})$ by Proposition 2 (iii). Hence, the sequence $\{y^n\}_{n=0}^{\infty}$ represents a sequence of vectors y^n that are monotonically decreasing with lower bound zero because of Proposition 2 (i). It follows that y^n converges to a limit $\bar{y} = \lim_{n\to\infty} y^n$. Since the transformation function T is continuous, see Proposition 2 (ii), this limit $\bar{y}$ is a fixed point of T, and thus Eq. (5) in Lemma 2 is satisfied by $O(\bar{x}) = \bar{y}$. It follows directly, from Lemma 1 that Eq. (6) yields a proportionally fair solution $\bar{x}$ to the divorced parents problem. ☐

Based on Theorem 1 we define the following fixed point iteration algorithm (FPA) for computing the proportionally fair solution.

Algorithm 1: Fixed point iteration algorithm (FPA).

1. **Initialization.** Let overage parent-child network $\mathscr{N} = (\mathscr{V}, \mathscr{E}, d, b)$ and initial parental overages $y^0 = d$ be given. Let $\epsilon > 0$ denote the tolerance level, and set $n := 0$.

2. **Iteration.** Use the transformation function T from Definition 4 to update the parental overages. That is, for every parent $i \in \mathscr{P}$, set

$$y_i^{n+1} := d_i \Big(1 + \sum_{j \in \mathscr{C}(i)} \frac{b_j}{\sum_{k \in \mathscr{P}(j)} y_k^n} \Big)^{-1}.$$

Update the iteration counter, set $n := n+1$.

3. **Termination.** Stop if $\sum\limits_{i \in \mathscr{P}} y_i^n / (\sum\limits_{i \in \mathscr{P}} d_i - \sum\limits_{j \in \mathscr{C}} b_j) \leq 1 + \epsilon$, and use Eq. (6) to compute the proportional fair solution:

$$\bar{x}_{ij}^n = \frac{y_i^n}{\sum_{k \in \mathscr{P}(j)} y_k^n} b_j, \qquad \forall j \in \mathscr{C}, i \in \mathscr{P}(j). \tag{7}$$

In the FPA of Algorithm 1, the y-values, representing parental overages, decrease monotonically, and thus the corresponding parental contributions increase. Eventually, these parental contributions cover the needs of all children in the network. This is what the stopping criterion in Algorithm 1 guarantees.

REMARK 1. The smaller the threshold level ϵ in Algorithm 1, the better we expect $\bar{x}^n$ from Algorithm 1 to approximate $\bar{x}$. If ϵ is too large, however, it is possible that $\bar{x}^n \notin \mathscr{X}$. In that case, the tolerance level ϵ should be decreased and more iterations need to be carried out.

5. Convergence in case of iterative lawsuits

In the previous section we have developed a fixed-point iteration algorithm for computing the proportionally fair solution $\bar{x}$ of a parent-child network $\mathscr{N}$. Since this solution $\bar{x}$ is unique it allows judges to settle cases without discussion, even for large parent-child networks $\mathscr{N}$. However, in practice typically only two or three parents are involved in a case, and thus court rulings often only apply to parts of the parent-child network and not the entire network. Since a lawsuit for a subnetwork may increase the unfairness in other parts of the network, this typically leads to a *sequence of lawsuits*, each corresponding to different small subnetworks of the larger network $\mathscr{N}$. From a societal point of view it is desired that this sequence of lawsuits does not continue indefinitely, but (quickly) yields the proportionally fair solution $\bar{x}$ of the entire network. In this section we prove under very mild conditions that the parental contributions indeed converge to the unique proportionally fair solution $\bar{x}$ of Theorem 1.

Before we prove the main convergence result of this section, we first give a definition of a lawsuit and explain how it changes the parental contributions x.

DEFINITION 5. Consider an overage parent-child network $\mathscr{N} = (\mathscr{V}, \mathscr{E}, d, b)$. Then, we call $N = (V, E)$, with $V = P \cup C$, a *subnetwork* of $\mathscr{N}$ if N is connected and $P \subset \mathscr{P}$, $C \subset \mathscr{C}$, and $E \subset \mathscr{E}$. We write $N \subset \mathscr{N}$, and only consider subnetworks N that contain at least two parents and a common child.

DEFINITION 6. Consider an overage parent-child network $\mathscr{N}$ and let $N \subset \mathscr{N}$ be a subnetwork of $\mathscr{N}$. Then, a *lawsuit* $\mathscr{L}_N : \mathscr{X} \to \mathscr{X}$ on subnetwork N transforms the parental contributions $x \in \mathscr{X}$ according to

$$\mathscr{L}_N(x_{ij}) = \begin{cases} \bar{x}_{ij}(N, \bar{d}(x), \bar{b}), & \text{if } (i,j) \in E, \\ x_{ij}, & \text{if } (i,j) \notin E, \end{cases} \tag{8}$$

where $\bar{b}$ denotes the financial needs of the children in N, and $\bar{d}_i(x) := d_i - \sum_{j:(i,j)\in\mathscr{E}\setminus E} x_{ij}$, $i \in P$, represents the financial capacity of parent i in subnetwork N when all *current* obligations to all children *not* in N are subtracted. We call a lawsuit $\mathscr{L}_N$ *simple* whenever the subnetwork N consists of only a single child $j \in \mathscr{C}$ and its parents $\mathscr{P}(j)$. We let $\mathscr{S}$ denote the set of all simple lawsuits.

REMARK 2. Observe from Eq. (8) that a lawsuit $\mathscr{L}_N$ only alters parental contributions x_{ij} of parents and children in the subnetwork N.

LEMMA 3. *Let $\mathscr{N}$ be an overage parent-child network. Then,*

(i) The lawsuit $\mathscr{L}_N$ is continuous in x for every $N \subset \mathscr{N}$.

(ii) A lawsuit acts as a projection, i.e., $(\mathscr{L}_N \circ \mathscr{L}_N)(x) = (\mathscr{L}_N)(x)$ for all $x \in \mathscr{X}$ and $N \subset \mathscr{N}$.

(iii) Non-overlapping lawsuits can be interchanged, i.e., $(\mathscr{L}_{N_1} \circ \mathscr{L}_{N_2})(x) = (\mathscr{L}_{N_2} \circ \mathscr{L}_{N_1})(x)$ for all $x \in \mathscr{X}$ and $N_1, N_2 \subset \mathscr{N}$ with $N_1 \cap N_2 = \emptyset$.

Proof. The lawsuit $\mathscr{L}_N$ is continuous for $N \subset \mathscr{N}$ if and only if the proportionally fair solution $\bar{x}$ is continuous in d. The latter is true since T is continuous in y and d, and thus by the implicit function theorem, any fixed point $\bar{y}$ of T is continuous in d. It then follows directly from (6) that $\bar{x}$ is also continuous in d, proving (i).

To prove (ii) and (iii), observe that $\bar{d}(x)$ is determined by parental contributions x_{ij} of parents $i \in P$ and children $j \in \mathscr{C}\setminus C$. Moreover, lawsuit $\mathscr{L}_N$ only adjusts parental contributions x_{ij} of parents $i \in P$ and children $j \in C$. Thus, $\bar{d}(x) = \bar{d}(\mathscr{L}_N(x))$ for all $x \in \mathscr{X}$, and it follows directly from the definition of $\mathscr{L}_N$ that (ii) holds. Moreover, if N_1 and N_2 are disjoint, then for all $x \in \mathscr{X}$ we have $d_{N_1}(x) = d_{N_1}(\mathscr{L}_{N_2}(x))$ and $d_{N_2}(x) = d_{N_2}(\mathscr{L}_{N_1}(x))$, and thus (iii) also holds. □

Property (ii) in Lemma 3 shows that repeating the same lawsuit directly does not change the parental contributions. Moreover, Lemma 3 (iii) implies that when two lawsuits have no parents or children in common, then the order of the lawsuits does not matter.

The following proposition shows that the proportionally fair solution $\bar{x}$ is the only solution $x \in \mathscr{X}$ that is unaltered by *all* simple lawsuits $\mathscr{L}_N$ on $\mathscr{N}$.

PROPOSITION 3. *Consider an overage parent-child network $\mathscr{N}$. Then, $\mathscr{L}_N(x) = x$ for all simple lawsuits $\mathscr{L}_N \in \mathscr{S}$ if and only if $x = \bar{x}$.*

Proof. '⇐' Obviously, if $x=\bar{x}$, then $\mathscr{L}_N(x)=x$ for all $\mathscr{L}_N\in\mathscr{S}$.
'⇒' Let $x\in\mathscr{X}$ be given such that $\mathscr{L}_N(x)=x$ for all $\mathscr{L}_N\in\mathscr{S}$. We will show that x satisfies the proportionality conditions from Proposition 1 so that $x=\bar{x}$. To see this, consider child $j\in\mathscr{C}$ and its parents $\mathscr{P}(j)$. Since they correspond to a simple lawsuit $\mathscr{L}_N\in\mathscr{S}$ and since $\mathscr{L}_N(x)=x$, it holds that for every two parents $i,k\in P(j)$ of child j, we have $\frac{x_{ij}}{B_{ij}(x)}=\frac{x_{kj}}{B_{kj}(x)}=\delta_j$. □

Proposition 3 provides a stopping criterion for when nobody is better off by starting a simple case. At the same time, it shows that only in the proportionally fair solution this is true. In fact, this result does not only hold for simple cases, but lawsuits that involve subnetworks of $\mathscr{N}$ of any size.

DEFINITION 7. We call a collection of subnetworks $\{N_1,\ldots,N_K\}$ of $\mathscr{N}$ a *network cover* of $\mathscr{N}$ if every network N associated to a simple lawsuits is a subset of N_k for at least one $k=1,\ldots,K$.

Note that, in particular, the set $\mathscr{S}$ of all simple lawsuits yields a network cover.

COROLLARY 1. *Consider an overage parent-child network $\mathscr{N}$, and let $\{N_1,\ldots,N_K\}$ be a network cover of $\mathscr{N}$. Then, $\mathscr{L}_{N_k}(x)=x$ for all $k=1,\ldots,K$, if and only if $x=\bar{x}$.*

Proof. Analogous to Proposition 3. □

To prove that a sequence of parental contributions $\{x^n\}_{n=0}^{\infty}\subset\mathscr{X}$ with $x^n:=\mathscr{L}_{N_n}(x^{n-1})$, converges to the proportionally fair solution $\bar{x}$ for a sequence of lawsuits $\{\mathscr{L}_{N_n}\}_{n=1}^{\infty}$, we use a so-called *fairness function* $F:\mathscr{X}\to\mathbb{R}$. This function will act as a Lyapunov function in the sense that $F(\mathscr{L}_N(x))>F(x)$ if $\mathscr{L}_N(x)\neq x$. That is, the fairness increases after a relevant lawsuit. Part of Moulin and Sethuraman (2013, Eq 5) provides the function we need.

DEFINITION 8. Consider an overage parent-child network $\mathscr{N}$. We define the *fairness function* $F:\mathscr{X}\to\mathbb{R}$ of $\mathscr{N}$ as

$$F(x)=\sum_{i\in\mathscr{P}}\sum_{j\in\mathscr{C}(i)}\operatorname{Ln}\left(x_{ij}\right)+\sum_{i\in\mathscr{P}}\operatorname{Ln}\left(O_i(x)\right),\qquad x\in\mathscr{X},\tag{9}$$

where $\operatorname{Ln}:\mathbb{R}^+\to\mathbb{R}$ is defined as $\operatorname{Ln}(y)=y-y\log(y)$ with the convention $\operatorname{Ln}(0)=0$.

LEMMA 4. *The fairness function F satisfies the following properties.*

(i) The function F is strictly concave.

(ii) The function F is bounded on $\mathscr{X}$.

(iii) The proportionally fair solution $\bar{x}$ is the unique global maximizer of F.

Proof. See Moulin and Sethuraman (2013).

Next, we show that the fairness function strictly increases after any regular lawsuit.

DEFINITION 9. For every $x\in\mathscr{X}$ and $N\subset\mathscr{N}$ we call a lawsuit $\mathscr{L}_N$ a *regular lawsuit*, if $\mathscr{L}_N(x)\neq x$.

PROPOSITION 4. *Consider an overage parent-child network $\mathscr{N}$, and let $x\in\mathscr{X}$ be given. Then, $F(\mathscr{L}_N(x))>F(x)$ if $\mathscr{L}_N$ is a regular lawsuit, and $F(\mathscr{L}_N(x))=F(x)$, otherwise.*

Proof. If a lawsuit $\mathscr{L}_N$ is not regular, then $\mathscr{L}_N(x) = x$, and thus $F(\mathscr{L}_N(x)) = F(x)$. A regular lawsuit $\mathscr{L}_N$, on the other hand, only alters the parental contributions x_{ij} of parents $i \in P$ and children $j \in C(i)$, and thus we can rewrite the difference $F(\mathscr{L}_N(x)) - F(x)$ as $G(\mathscr{L}_N(x)) - G(x)$ with

$$G(x) = \sum_{i\in P}\sum_{j\in C(i)} \operatorname{Ln}\left(x_{ij}\right) + \sum_{i\in P} \operatorname{Ln}\left(d_i - \sum_{j\in\mathscr{C}(i)\setminus C(i)} x_{ij} - \sum_{j\in C(i)} x_{ij}\right), \qquad x \in \mathscr{X}. \tag{10}$$

Here, $\sum_{j\in\mathscr{C}(i)\setminus C(i)} \mathscr{L}_N(x)_{ij} = \sum_{j\in\mathscr{C}(i)\setminus C(i)} x_{ij}$ for all $i \in P$ since $\mathscr{L}_N(x)_{ij} = x_{ij}$ for all $j \notin C(i)$. Interestingly, the function G corresponds to the fairness function $F_{\hat{\mathscr{N}}}$ of the smaller network $\hat{\mathscr{N}} = (N, d_N(x), b_N)$. Moreover, $\mathscr{L}_N(x)$ restricted to N corresponds to the proportionally fair solution $\bar{x}(\hat{\mathscr{N}})$ of $\hat{\mathscr{N}}$. Since by Lemma 4 (iii), $\bar{x}(\hat{\mathscr{N}})$ is the unique global maximizer of $F_{\hat{\mathscr{N}}}$, and since the lawsuit $\mathscr{L}_N$ is regular, it follows that $G(\mathscr{L}_N(x)) - G(x) > 0$, and thus $F(\mathscr{L}_N(x)) > F(x)$ if $\mathscr{L}_N(x) \neq x$. □

Before we prove the convergence of $\{x^n\}_{n=0}^{\infty}$, we first need some mild assumptions on the sequence of lawsuits $\{\mathscr{L}_{N_n}\}_{n=1}^{\infty}$. For example, if we only repeat the same lawsuit, i.e., if $\mathscr{L}_{N_n} = \mathscr{L}_N$ for all $n \in \mathbb{N}$, then $x^n = \mathscr{L}_N(x^0)$ for all $n \in \mathbb{N}$ by Lemma 3, and this sequence does not necessarily converge to the proportionally fair solution $\bar{x}$. Similarly, if N_n does not contain parent $i \in \mathscr{P}$ for all $n \in \mathbb{N}$, then $x_{ij}^n = x_{ij}^0$ for all $j \in \mathscr{C}(i)$, and again the proportionally solutions may not be achieved. To exclude these trivial cases, we only consider *admissible* sequences of lawsuits, defined as follows.

DEFINITION 10. Consider an overage parent-child network $\mathscr{N}$. A *covering pattern* is a fixed and finite sequence $\Pi = (\hat{N}_1, \hat{N}_2, \ldots, \hat{N}_K)$ of sets $\hat{N}_i \subset \mathscr{N}$ such that $\{\hat{N}_i : i \leq K\}$ is a network cover. A sequence $\{\mathscr{L}_{N_n}\}_{n=1}^{\infty}$ of lawsuits is *admissible* when it contains a covering pattern Π infinitely often.

Now we are ready to prove that under very mild assumptions the sequence $\{x^n\}_{n=0}^{\infty} \subset \mathscr{X}$ with $x^n = \mathscr{L}_{N_n}(x^{n-1})$, converges to $\bar{x}$ when cases $\{\mathscr{L}_{N_n}\}_{n=1}^{\infty}$ are iteratively settled.

THEOREM 2. *Consider an overage parent-child network $\mathscr{N}$, and let $\{\mathscr{L}_{N_n}\}_{n=1}^{\infty}$ and $x^0 \in \mathscr{X}$ be given. If the sequence $\{\mathscr{L}_{N_n}\}_{n=1}^{\infty}$ of lawsuits is admissible, then the sequence of parental contributions $\{x^n\}_{n=0}^{\infty} \subset \mathscr{X}$, defined by $x^n = \mathscr{L}_{N_n}(x^{n-1})$ for every $n \in \mathbb{N}$, converges to the proportionally fair solution $\bar{x}$.*

Proof. Consider the sequence $\{F(x^n)\}_{n=0}^{\infty} \subset \mathbb{R}$. Since by Proposition 4 it holds that $F(\mathscr{L}(x)) \geq F(x)$ for all $x \in \mathscr{X}$ and $N \subset \mathscr{N}$, it follows that the sequence $\{F(x^n)\}_{n=0}^{\infty}$ is non-decreasing, i.e., $F(x^n) \geq F(x^{n-1})$ for all $n \in \mathbb{N}$. Moreover, since $F(x^n)$ is bounded from above by $F(\bar{x})$, see Lemma 4 (iii), it follows that $F(x^n)$ converges to some limit $L \leq F(\bar{x})$.

Next, we will construct a subsequence of $\{x^n\}_{n=0}^{\infty}$ that converges to $\bar{x}$. For this purpose, let the subsequence $\{x^{n_l}\}_{l=0}^{\infty}$ be such that x^{n_l} is an element of $\{x^n\}_{n=0}^{\infty}$ that appears just *before* the first case $\hat{N}_1$ of some occurrence of the covering pattern Π. Such a subsequence exists since $\{\mathscr{L}_{N_n}\}_{n=1}^{\infty}$ is admissible, and thus the pattern Π occurs infinitely often. Moreover, since the sequence $\{x^{n_l}\}_{l=0}^{\infty} \subset \mathscr{X}$ is defined on a compact space, it has a converging subsequence by the theorem of Bolzano-Weierstrass. Hence, without loss of generality we assume that the elements of $\{x^{n_l}\}_{l=0}^{\infty}$ are selected such that $\{x^{n_l}\}_{l=0}^{\infty}$ is converging, and we call the limit $\hat{x}$. Using the subsequence $\{x^{n_l}\}_{l=0}^{\infty}$ we construct a second subsequence $\{z^l\}_{l=0}^{\infty}$ of $\{x^n\}_{n=0}^{\infty}$, by defining $z^l = \mathscr{L}_{\hat{N}_K} \circ \cdots \circ \mathscr{L}_{\hat{N}_1}(x^{n_l})$ for $l = 0, \ldots, \infty$. That is, z^l is the element of $\{x^n\}_{n=0}^{\infty}$ *after* the pattern Π is applied to x^{n_l}. Moreover, since $\mathscr{L}_{\hat{N}_K} \circ \cdots \circ \mathscr{L}_{\hat{N}_1}$ is continuous by Lemma 3 (i), the subsequence $\{z^l\}_{l=0}^{\infty}$ converges to a limit called $\hat{z}$ for which

$$\hat{z} := \lim_{l\to\infty} z^l = \lim_{l\to\infty} \mathscr{L}_{\hat{N}_K} \circ \cdots \circ \mathscr{L}_{\hat{N}_1}(x^{n_l}) = \mathscr{L}_{\hat{N}_K} \circ \cdots \circ \mathscr{L}_{\hat{N}_1}(\hat{x}). \tag{11}$$

Now, since $\{x^{n_l}\}_{l=0}^{\infty}$ and $\{z_l\}_{l=0}^{\infty}$ are subsequences of $\{x^n\}_{n=0}^{\infty}$, it follows that $F(\hat{x}) = F(\hat{z}) = L$. Since F is continuous, and using Eq. (11), we have

$$F(\hat{x}) = F(\hat{z}) = F((\mathscr{L}_{\hat{N}_K} \circ \cdots \circ \mathscr{L}_{\hat{N}_1})(\hat{x})).$$

This equality shows that the fairness of $\hat{x} \in \mathscr{X}$ does not improve by applying cases $\mathscr{L}_{\hat{N}_1}, \ldots, \mathscr{L}_{\hat{N}_k}$ to $\hat{x}$. Since the pattern Π is a network cover, it follows from Corollary 1 that $\hat{z} = \hat{x} = \bar{x}$. Moreover, $L = F(\hat{x}) = F(\bar{x})$.

It remains to prove that $\{x^n\}_{n=0}^{\infty}$ converges to $\bar{x}$. Suppose that this is not the case. Then, there exists $\epsilon > 0$ such that for infinitely many $n \in \mathbb{N}$, we have $\|x^n - \bar{x}\| \geq \epsilon$. However, on $\{x \in \mathscr{X} : \|x - \bar{x}\| \geq \epsilon\}$, the fairness function F has a global maximum $F_\epsilon < F(\bar{x})$. This means that infinitely many often, $F(x^n) \leq F_\epsilon < F(\bar{x})$, contradicting that $F(x^n) \to F(\bar{x})$. Hence, $\{x^n\}_{n=0}^{\infty}$ converges to $\bar{x}$. □

The next corollary is an immediate consequence of Theorem 2.

COROLLARY 2. *Consider an overage parent-child network $\mathscr{N}$, and let $x^0 \in \mathscr{X}$ be given. Suppose that a rule π selects each element $\hat{N}_i$ of a network cover with positive probability as the nth lawsuit of a sequence of lawsuits $\{\mathscr{L}_{N_n}\}_{n=1}^{\infty}$ for every $n \in \mathbb{N}$. Then, the sequence of parental contributions $\{x^n\}_{n=0}^{\infty}$ obtained under π converges to the proportionally fair solution $\bar{x}$ almost surely.*

Proof. The randomized policy π produces the covering pattern $\Pi = (\hat{N}_1, \ldots, \hat{N}_K)$ with positive probability, and thus this covering pattern almost surely occurs infinitely often in a sequence of lawsuits. Such sequences being admissible, the convergence of $\{x^n\}_{n=0}^{\infty}$ follows directly from Theorem 2. □

Theorem 2 and Corollary 2 show that the parental contributions based on a sequence of lawsuits on subnetworks of the larger parent-child network converges to the proportionally fair solution $\bar{x}$ of the complete parent-child network under very mild assumptions. This result is not only desirable from a societal point of view, but also provide us with a very simple and elegant procedure for computing the proportionally fair solution to a given parent-child network: just iteratively apply the effect of lawsuits on smaller subnetworks. In particular, we propose to apply simple lawsuits only, since these are easy to understand and the proportionally fair solution can be determined analytically for these simple lawsuits, see, e.g., Example 1.

Algorithm 2: Iteratively apply simple lawsuits.

1. **Initialization.** Let overage parent-child network $\mathcal{N} = (\mathcal{V}, \mathcal{E}, d, b)$ and initial parental contributions $x^0 \in \mathcal{X}$ be given. Let $\epsilon > 0$ denote the tolerance level and set $n := 0$.
2. **Iteration.** Select a child $\bar{j} \in \mathcal{C}$ and apply the effects of the simple lawsuit $\mathcal{L}_N \in \mathcal{S}$ corresponding to child $\bar{j}$. That is, update the parental contributions according to

$$x_{ij}^{n+1} = \begin{cases} B_{ij}(x^n) \dfrac{b_j}{\sum_{k \in P(j)} B_{kj}(x^n)}, & \text{if } j = \bar{j} \text{ and } i \in \mathcal{P}(\bar{j}), \\ x_{ij}^n, & \text{otherwise.} \end{cases}$$

   Update the iteration counter, set $n := n + 1$.
3. **Termination.** Stop if $\max_{i \in \mathcal{P}, j \in \mathcal{C}(j)} |\mathcal{L}_N(x^n)_{ij} - x_{ij}^n| < \epsilon$, for all $\mathcal{L}_N \in \mathcal{S}$.

In Section 6, we compare several alternatives of Algorithm 2 that differ in the order in which simple lawsuits are carried out, including randomly selecting simple lawsuits and repeatedly applying the simple lawsuit with the largest impact. First, however, we revisit the actual court case from Example 3 to illustrate the effectiveness of iteratively applying simple lawsuits.

EXAMPLE 4 (EXAMPLE 3 REVISITED). Consider the case from Example 3 with the actual court rulings as presented in Table 1. We iteratively apply a few simple lawsuits to the solution provided by court and compare the result with the proportionally fair solution $\bar{x}$, also given in Table 1. In fact, in Fig. 2 we show the differences between the overages of the parents for the current solution and those of the proportionally fair solution. To obtain these differences, first the simple lawsuit corresponding to child AC is carried out, and next the ones corresponding to CD, EF, and AB, in this order. We observe from Fig. 2 that the large differences between the current solution and the proportionally fair solution disappear fast after only a few simple lawsuits.

6. Numerical experiments

In this section we compare various methods to compute the proportionally fair solution $\bar{x}$ of a parent-child overage network $\mathcal{N}$. All computations are carried out on a desktop computer with i7 core and 16GB internal memory using Python 3.10.

6.1. Experimental design

We consider several different methods to compute the proportionally fair solution $\bar{x}$. The first is a benchmark method, referred to as CO, in which we use convex optimization to maximize the fairness function F from Definition 8 under the max-flow constraints $\sum_{i \in \mathcal{P}(j)} x_{ij} = b_j$ for all $j \in \mathcal{C}$. In particular, we use the state-of-the-art ECOS solver of the CVXPY library (Diamond and Boyd 2016, Agrawal et al. 2018). The second method is the fixed-point iteration algorithm (FPA) described in Algorithm 1, and finally we consider four versions of Algorithm 2 that differ in how simple lawsuits are iteratively selected. In the first, called IT Random, we select simple lawsuits randomly

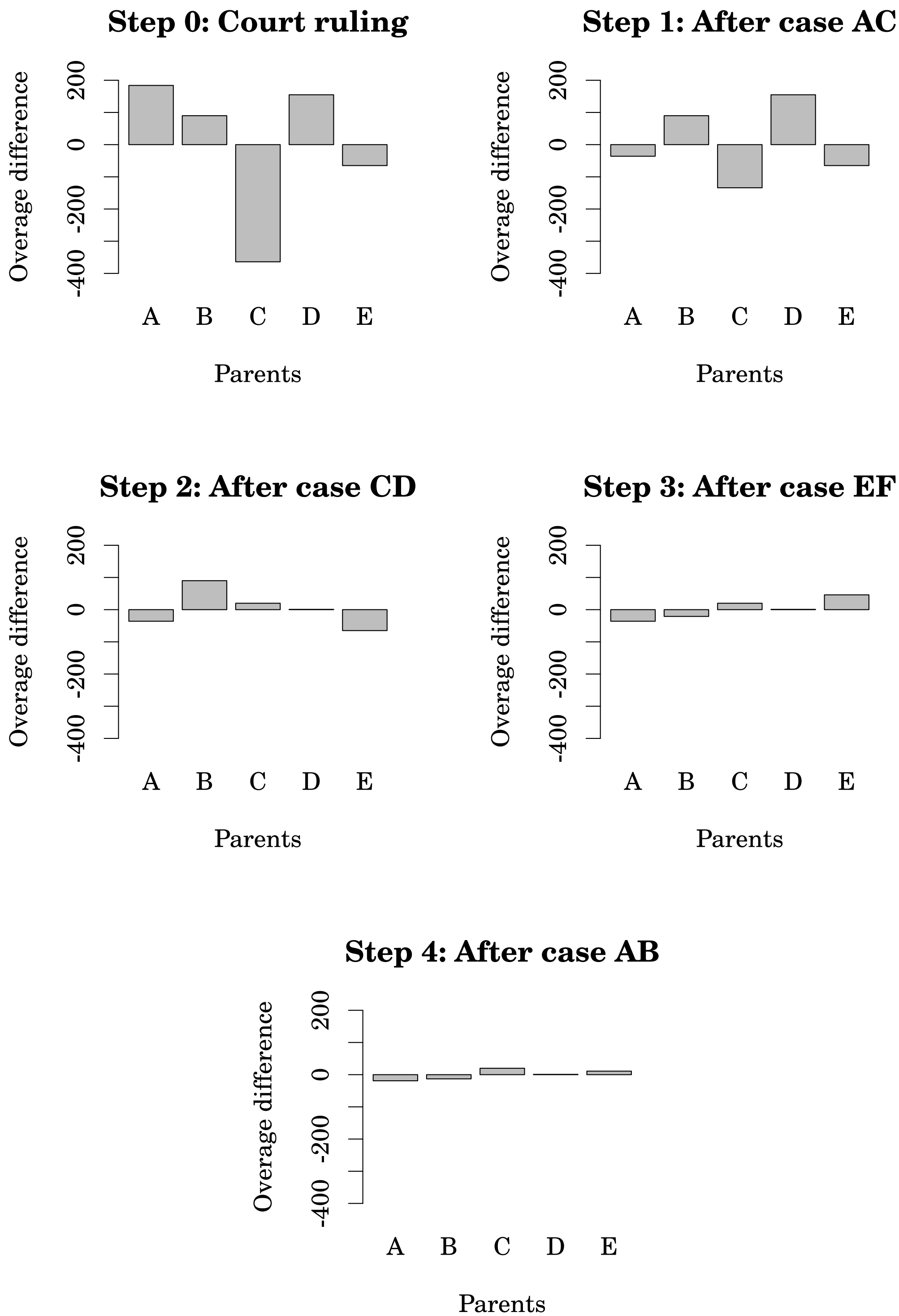

Figure 2 The differences in parental overages compared to $O(\bar{x})$ in Example 3 after several simple lawsuits.

with equal probability. In the second, called IT Wave, we carry out the $|\mathscr{S}|$ simple lawsuits in a pre-determined order, and then immediately afterwards in reverse order. This process is repeated with the same fixed order of $|\mathscr{S}|$ simple lawsuits. In the third version of Algorithm 2, called IT Best Update, we iteratively apply the simple lawsuit $\mathscr{L}_N \in \mathscr{S}$ with the largest impact in parental contribution, i.e., $\arg\max_{\mathscr{L}_N \in \mathscr{S}}\{|\mathscr{L}_N(x)) - x|\}$. The fourth method is motivated by the fact that it may be computationally expensive to determine the simple lawsuit $\mathscr{L}_N$ with the largest impact. Hence, in the final method, called IT Sorted Update, we first sort all $|\mathscr{S}|$ simple lawsuits based on their impact, and then carry out all lawsuits in this order, before computing their impact again and resorting the simple lawsuits. Table 2 summarizes these solution methods.

Table 2 A description of the solution methods that we use in our numerical experiments.

Abbreviation	Description
CO	Convex optimization method that maximizes the fairness function F over $\mathscr{X}$.
FPA	The fixed-point iteration algorithm of Theorem 1.
IT Random	Algorithm 2 with simple lawsuits randomly selected with equal probability.
IT Wave	Algorithm 2 in which simple lawsuits are applied in a fixed order and reversed.
IT Best Update	Algorithm 2 with simple lawsuits with the largest impact iteratively applied.
IT Sorted Update	Algorithm 2 where simple lawsuits are sorted on impact and then applied.

We note that we use a different stopping criterion for the FPA and the four IT methods than described in Algorithms 1 and 2, to be able to better compare these methods. We let $M := F(\bar{x}_{\text{FPA}})$ denote the value of the fairness function F for the parental contributions $\bar{x}_{\text{FPA}}$ obtained by Algorithm 1 with tolerance level $\epsilon = 10^{-2}$. Next, we run both FPA and the IT methods, until $(M - F(x^n))/|M| \le 10^{-5}$, and we use the same relative tolerance level of 10^{-5} for CO.

However, since it is computationally expensive to compute $F(x^n)$, we only check the termination criterion in the FPA when $\max_i\{y_i^n\}/\max_i\{y_i^{n-1}\} \le 1 + 10^{-6}$. Similarly, for IT Random and IT Wave we check the termination criterion only after every $2|\mathscr{S}|$ simple lawsuits, and for IT Sorted Update after $|\mathscr{S}|$ simple lawsuits. In contrast, for IT Best Update the value of $F(x^n)$ is known after every iteration since $F(\mathscr{L}_N(x^n))$ needs to be computed for all simple cases $\mathscr{L}_N$, to determine the simple case with the largest impact. Interestingly, for this algorithm the value of the fairness function $F(x^{n+1})$ can be updated after a simple lawsuit without computing the sum over all parental contributions x^{n+1} and parental overages $O(x^{n+1})$. Specifically, suppose there is a simple lawsuit between parents corresponding to child j. Then, the increase in fairness $\Delta F := F(x^{n+1}) - F(x^n)$ is given by

$$\Delta F = \sum_{i \in \mathscr{P}(j)} \Big(\operatorname{Ln} x_{ij}^{n+1} + \operatorname{Ln} O_i(x^{n+1}) - \operatorname{Ln} x_{ij}^{n} - \operatorname{Ln} O_i(x^n) \Big).$$

We apply these methods to three classes of randomly generated instances with different topologies. Namely, networks with topologies that are randomly generated, and networks with both a small and large diameter, called *star networks* and *linear networks*, respectively, see Fig. 3. Here, the diameter of a parent-child network is defined as the length of the largest shortest path between any two parents in the undirected version of $\mathcal{N}$. Below we describe in more detail how all instances are generated.

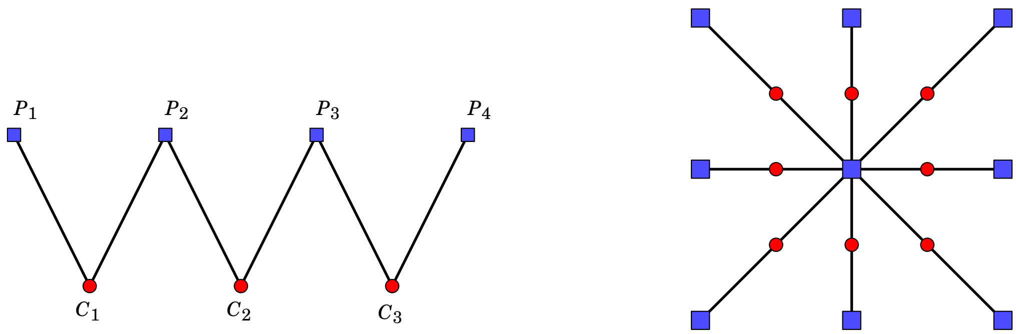

Figure 3 Two examples of parent-child networks with parents represented as squares and children as circles. The left graph is an example of a linear network with 4 parents and 3 children; the right of a star network with 9 parents and 8 children. In the star network, parent 0 is in the middle and child $i = 1,\ldots,8$, receives an allowance from parent 0 and parent i, respectively.

Star networks have one parent in the middle, called parent 0, which is connected to all other parents $i = 1,\ldots,|\mathcal{P}|-1$, via a common child also labeled as i, for all $i \in \mathcal{C}$. Hence, each child $i \in \mathcal{C}$ has two parents: parent 0 and parent i. In *linear networks*, on the other hand, each parent $i = 2,\ldots|\mathcal{P}|-1$ has two children, conveniently called $i-1$ and i, and each child $i \in \mathcal{C}$ has two parents, namely i and $i+1$. Parents 1 and $|\mathcal{P}|$ both only have a single child, namely child 1 and $|\mathcal{P}|-1$, respectively. See Fig. 3 for an example of a linear network and a star network. Finally, the third class of networks has a randomly generated topology. We start with a linear network with $|\mathcal{P}|$ parents and $|\mathcal{P}|-1$ children to ensure that the network is connected. Then, for every parent $i \in \mathcal{P}$ we randomly select ζ_i children from $\mathcal{C}$, with ζ_i discrete uniformly distributed on $\{0,1,\ldots,6\}$, and we make parent i financially responsible for these ζ_i children and those dictated by the linear network if not already selected.

For every type of network, we let b_j be independently and uniformly distributed on $[0.5, 1.5]$ for every $j \in \mathcal{C}$. Next, the financial capacities d of the parents are determined as follows. For every $i \in \mathcal{P}$, let d_i equal to

$$d_i = \xi_i \sum_{j \in \mathcal{C}(i)} \frac{b_j}{|\mathcal{P}(j)|},$$

where ξ_i is uniformly distributed on the interval $(1,5)$. Note that the parameters are selected in such a way that we always obtain an overage network. Moreover, note that when $\xi_i = 1$ for all $i \in \mathcal{P}$, then the total capacities of the parents equal the total needs of the children.

6.2. Numerical results

We generate networks of various sizes $|\mathscr{P}|$. For every type of network and size, we randomly generate 10 different problem instances and we apply the solution methods summarized in Table 2 to these instances. For each solution method, we report the mean running time and, if applicable, the mean number of iterations over these problem instances.

Table 3 Average running times in milliseconds for the solution methods from Table 2 for small parent-child networks. In between brackets the average number of iterations for the FPA and the average number of simple lawsuits carried out for the IT methods are reported. The running time of fastest algorithm per network is indicated with bold face. We report OOT if a solution method exceeds 5 minutes on at least one instance.

Size	Network	FPA	IT Wave	IT Random	IT Sorted	IT Best	CO
10	star	2 (9)	**1** (45)	2 (88)	2 (30)	6 (24)	80
10	linear	2 (6)	**1** (43)	1 (72)	1 (28)	5 (21)	81
10	random	**2** (5)	2 (43)	2 (83)	2 (29)	8 (25)	66
30	star	**4** (12)	4 (157)	7 (290)	5 (102)	37 (71)	762
30	linear	3 (7)	**2** (157)	4 (273)	4 (104)	26 (69)	725
30	random	**3** (5)	4 (162)	7 (278)	6 (87)	50 (78)	707
100	star	**12** (14)	101 (5128)	30 (1049)	33 (634)	522 (246)	13402
100	linear	8 (9)	**7** (594)	10 (1049)	9 (337)	180 (230)	13383
100	random	**9** (6)	12 (554)	19 (1129)	14 (317)	383 (266)	12888
300	star	**38** (18)	2227 (49335)	160 (3349)	290 (2990)	10777 (750)	OOT
300	linear	18 (8)	**14** (1794)	28 (3648)	24 (1196)	1542 (710)	OOT
300	random	**19** (5)	29 (1794)	52 (3528)	37 (897)	3562 (801)	OOT

Table 3 shows the running times (in milliseconds) of the solution methods for relatively small networks of size $|\mathscr{P}| = 10, 30, 100$, and 300 parents, respectively. If a method exceeds 5 minutes on at least one of these instances then we report “out of time (OOT)”. It is clear from Table 3 that the method CO is about one or two orders of magnitude slower than the other methods. For $|\mathscr{P}| =$ 300, CO is even out of time, exceeding 5 minutes, whereas some of the other methods finish in milliseconds. We note that all running times reported in Table 3 are ‘wall clock times’, even though for CO we could distinguish between setup time and solving time. We do not make this distinction to keep a fair comparison between all methods since the other methods do not have setup and solving times. Moreover, for completeness we should mention that CO is compiled C++ code, whereas the other algorithms run in pure Python. Thus, it should be clear from Table 3 that CO is significantly outperformed by both FPA and the IT methods. This is not surprising since CO is a general-purpose convex optimization solver, whereas the other methods are tailor-made for the DPP.

What may be surprising is that the method IT Best Update also performs poor, with running times much larger than FPA and the other IT methods. This can be explained by the fact that it takes too much time searching for the simple lawsuit with the largest impact. Indeed, as can be seen in

Table 3, the actual number of simple lawsuits that are carried out is substantially smaller for IT Best Update than for the other IT methods. However, for every simple lawsuit carried out we also investigate the impact of $|\mathscr{P}|-2$ alternative ones, which explains the large running times.

The running times of the method IT Sorted Update, however, are substantially smaller than those of IT Best Update since in this method we only compute the impact of each simple lawsuit after applying $|\mathscr{S}|$ of them. In this way, we carry out slightly more actual lawsuits but need less time to identify which simple lawsuits are promising ones. Compared with IT Wave and IT Random we indeed require less simple lawsuits, however, in terms of running time IT Sorted Update does not clearly outperform these methods. We conclude that any numerical effort invested in computing the impact of a simple lawsuit is better invested in actually carrying out the lawsuit. This is in line with Proposition 4 since the fairness, measured by F, always increases for any regular lawsuit.

Another interesting observation from Table 3 is that the performance of the methods depends on the topology of the network. For example, the IT Wave algorithm performs well for linear networks since we carry out simple lawsuits in a fixed predetermined order, in our case from child 1 to $|\mathscr{P}|-1$ and next from $|\mathscr{P}|-1$ back to 1, i.e., 'from left to right' and next 'from right to left' in the linear network, so that changes in parental contributions are propagated efficiently through the network. For random and star networks, however, this order of simple lawsuits is not necessarily the best, explaining why IT Random is better on those networks whereas IT Wave is best on linear networks. Moreover, note that all methods require more time for solving star networks than linear or random networks.

Overall, we conclude that FPA is fastest. It outperforms all IT methods on all types of network except for IT Wave on linear networks. This is according to expectation since the IT Wave algorithm is targeted at linear networks, whereas FPA cannot benefit from any special topology of the graph. The general superior performance of FPA cannot directly be explained by the number of iterations reported in Table 3 since an iteration in FPA requires computations for all parents and children in the network, whereas an iteration in the IT methods corresponds to a simple lawsuit and thus only requires computation for a single child and its parents. Nevertheless, the number of iterations for FPA seems to grow less fast than the number of simple lawsuits required for the IT methods.

For practical purposes, i.e., for computing the proportionally fair solution for an actual court case in which typically much less than 300 parents are involved, IT Wave, IT Random, and IT Sorted Update will be very efficient and run within a second. They have the advantage that they are easy to understand, and thus will be more easily accepted by parents, lawyers, and judges.

For applications of the general bipartite rationing problems other than the DPP, we recommend FPA since it is the fastest and scales favorably as a function of the size of the problem. This is confirmed in Fig. 4, from which we observe that the running times of FPA scale approximately linearly in the number of parents. This is because the number of iterations required by FPA remains small whereas the time required for carrying out a single iterations scales linearly in the size of the network. Fig. 4 shows that we can solve bipartite rationing problems with up to 10^5 nodes in less than a minute.

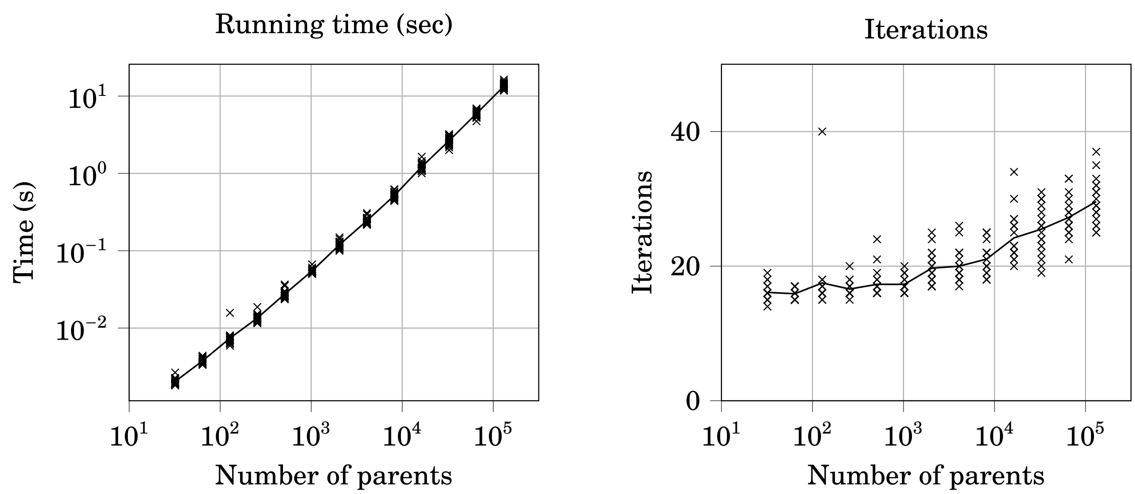

Figure 4 Running times (left) and number of iterations (right) for the FPA on large random networks. In the left graph both axes are on a log scale; in the right graph only the horizontal one. Each cross represents the result of a single instance; we ran 20 instances for each network size.

7. Summary and discussion

We consider the divorced-parents problem (DPP), in which divorced parents have to cover the financial needs of the children for which they are financially responsible. We derive mathematical properties that uniquely define *proportionally fair* parental contributions, also for complex cases involving children from previous marriages and parents remarrying parents that already have children. We show that the DPP can be represented as a general bipartite rationing problem, and design fast and efficient algorithms to compute the proportionally fair parental contributions. In addition, since lawsuits are often held between two or a few parents, even if the number of ex-partners involved is larger, we prove that iteratively applying such smaller lawsuits eventually leads to the same proportionally fair parental contributions as a single lawsuit with all relevant parents involved.

Numerical experiments show that our newly developed algorithms outperform standard convex optimization methods for the DPP, and thus also for the general bipartite rationing problem. In particular, the fixed point iteration algorithm FPA from Algorithm 1 is capable of solving large problem instances with hundred thousand nodes within less than a minute. Moreover, we show that it is also possible to efficiently compute the proportionally fair parental contributions for the DPP by iteratively applying so-called simple lawsuits, involving only a single child and all its parents. This result has an important off-spin with regard to the acceptance of the solution in practice, since each such simple lawsuit can be computed by hand, and hence is easy to understand by parents, legal counselors, and judges. Therefore, rather than having to rely on a 'black-box' method such as a convex optimization solver, the solution results from a repetition of simple numerical steps.

Finally, our method has the potential for significant practical impact. Besides being insightful, it provides a unique solution to any dispute, thereby removing the legal inequality perceived by parents. Moreover, it can considerably reduce the workload of courts, mediators, and lawyers, since when parents agree on the parental responsibilities, parental capacities and children needs they can use our algorithms to compute the unique solution without intervention of the court.

References

A. Agrawal, R. Verschueren, S. Diamond, and S. Boyd. A rewriting system for convex optimization problems. *Journal of Control and Decision*, 5(1):42–60, 2018.

R.K. Ahuja, T.L. Magnanti, and J.B. Orlin. *Network Flows: Theory, Algorithms and Applications*. Prentice Hall, 1993.

O. Bochet, R. Ilkılıç, H. Moulin, and J. Sethuraman. Balancing supply and demand under bilateral constraints. *Theoretical Economics*, 7:395–423, 2012.

S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

Case:767. ECLI:NL:RBOVE:2016:767. `https://uitspraken.rechtspraak.nl/inziendocument?id=ECLI:NL:RBOVE:2016:767`, 2016. Accessed: 2021-09-28.

S. Diamond and S. Boyd. CVXPY: A Python-embedded modeling language for convex optimization. *Journal of Machine Learning Research*, 17(83):1–5, 2016.

M. Jonker, J. Wijngaard, and N.D. Van Foreest. Over proportioneel verdelen van draagkracht in samengestelde gezinnen bij de bepaling van kinderalimentatie. *EB tijdschrijft voor scheidingsrecht*, 7/8:141–149, 2020.

F.P. Kelly, A.K. Maulloo, and D.K.H. Tan. Rate control for communication networks: shadow prices, proportional fairness and stability. *Journal of the Operational Research Society*, 49:237–252, 1998.

H. Moulin and J. Sethuraman. The bipartite rationing problem. *Operations Research*, 61(5):1087–1100, 2013.

N.S. Walton. Proportional fairness and its relationship with multi-class queueing networks. *The Annals of Applied Probability*, 19:2301–2333, 2009.

W. Wang, S.T. Maguluri, R. Srikant, and L. Ying. Heavy-traffic insensitive bounds for weighted proportionally fair bandwidth sharing policies. *Mathematics of Operations Research*, 2022. doi: https://doi.org/10.1287/moor.2021.1225.

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2022010-OPERA: Romeijnders, W., N.D. van Foreest and J. Wijngaard, On Proportionally Fair Solutions for the Divorced-Parents Problem www.rug.nl/feb